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March 20, 2011

Abstract

We consider a Ginzburg-Landau type energy with a piecewise constant pinning term a in the potential $(a^2 - |u|^2)^2$. The function a is different from 1 only on finitely many disjoint domains, called the *pinning domains*. These pinning domains model small impurities in a homogeneous superconductor and shrink to single points in the limit $\varepsilon \to 0$; here, ε is the inverse of the Ginzburg-Landau parameter. We study the energy minimization in a smooth simply connected domain $\Omega \subset \mathbb{C}$ with Dirichlet boundary condition g on $\partial\Omega$, with topological degree $\deg_{\partial\Omega}(g) = d > 0$. Our main result is that, for small ε, minimizers have d distinct zeros (vortices) which are inside the pinning domains and they have a degree equal to 1. The question of finding the locations of the pinning domains with vortices is reduced to a discrete minimization problem for a finite-dimensional functional of renormalized energy. We also find the position of the vortices inside the pinning domains and show that, asymptotically, this position is determined by *local renormalized energy* which does not depend on the external boundary conditions.

Contents

[1]AMS Subject Classification: 49K20, 35J66, 35J50.

1 Introduction and main results

In this work we study the minimizers of the Ginzburg-Landau type functional

$$E_{\varepsilon,\delta}(u) = \frac{1}{2}\int_{\Omega}\left\{|\nabla u|^2 + \frac{1}{2\varepsilon^2}(a_\delta^2 - |u|^2)^2\right\}, \tag{1}$$

where $\Omega \subset \mathbb{C}$ is a bounded, smooth, simply connected domain, ε is a positive parameter (the inverse of the Ginzburg-Landau parameter $\kappa = 1/\varepsilon$), $\delta = \delta(\varepsilon) > 0$ is a geometric parameter and u is a complex-valued map. In order to define the function a_δ, we need to introduce the notion of a *pinning domain.*

Fix $M \in \mathbb{N}^*$ points $a_1, ..., a_M \in \Omega$. Let ω be an open subset such that $\overline{\omega} \subset B(0,1)$ and $0 \in \omega$. For $1 \le i \le M$ and for all $\delta > 0$ denote $\omega_\delta^i := a_i + \delta \cdot \omega$, *i.e.* the set ω scaled by δ and centered at a_i.

Definition. The set $\omega_\delta := \cup_{i=1}^M \omega_\delta^i$ is called a *pinning domain.*

For example, if $\omega = B(0, \frac{1}{2})$, then the pinning domain is $\omega_\delta = \cup_{i=1}^M B(a_i, \frac{\delta}{2})$.

We now define $a_\delta : \Omega \to \{b, 1\}$, $b \in (0,1)$ as:

$$a_\delta(x) = \begin{cases} b & \text{if } x \in \omega_\delta \\ 1 & \text{if } x \in \Omega \setminus \omega_\delta \end{cases}. \tag{2}$$

The functionals of this type arise in models of superconductivity for composite superconductors. The experimental pictures suggest nearly 2D structure of parallel vortex tubes ([25], Fig I.4). Therefore, the domain Ω can be viewed as a cross-section of a multifilamentary wire with a number of thin superconducting filaments. Such multifilamentary wires are widely used in industry, including magnetic energy-storing devices, transformers and power generators [17], [16].

Another important practical issue in modeling superconductivity is to decrease the energy dissipation in superconductors. Here, the dissipation occurs due to currents associated with the motion of vortices ([21], [6]). This dissipation as well the thermomagnetic stability can be improved by *pinning* ("fixing the positions") of vortices. This, in turn, can be done by introducing impurities or inclusions in the superconductor. In the functional (1) the set ω_δ models the set of small impurities in a homogeneous superconductor. The size of the impurities in our model is characterized by the geometric parameter δ which goes to zero together with the material parameter ε. We assume henceforth that

$$\frac{|\ln\delta(\varepsilon)|^3}{|\ln\varepsilon|}\to 0. \tag{H}$$

Essentially, this condition means that δ is much larger than ε on the logarithmic scale. For example, if $\varepsilon = 2^{-j}$ and $\delta(\varepsilon) = 2^{-k(j)}$, then (H) implies that $\dfrac{k(j)^3}{j}\to 0$.

Notation. In what follows:

- We consider a sequence $\varepsilon_n \downarrow 0$ and we write ε instead of ε_n; the dependence of ε on n is implicit.
- We simply write δ (instead of $\delta(\varepsilon)$); the dependence of δ on ε is implicit.

We study the minimization problem for the functional (1) in the class

$$H^1_g := \{u\in H^1(\Omega,\mathbb{C})\,|\,\mathrm{tr}_{\partial\Omega}u = g\}, \tag{3}$$

where $g\in C^\infty(\partial\Omega,\mathbb{S}^1)$ is such that $\deg_{\partial\Omega}(g) = d > 0$. Recall that the degree (winding number) of g is defined as

$$\deg_{\partial\Omega}(g) := \frac{1}{2\pi}\int_{\partial\Omega} g\times\partial_\tau g\,\mathrm{d}\tau.$$

Here "$\times$" stands for the vectorial product in $\mathbb{C}$, *i.e.* $z_1\times z_2 = \mathrm{Im}(\overline{z_1}z_2)$, $z_1, z_2\in\mathbb{C}$, and ∂_τ is the tangential derivative. The degree is an integer, and the condition $\deg_{\partial\Omega}(u) = d > 0$, $u\in H^1(\Omega,\mathbb{C})$ implies that u must have at least d zeros (counting multiplicity) inside Ω. The properties of the topological degree can be found, *e.g.*, in [13] or [8].

Minimization problems for Ginzburg-Landau type functionals have been extensively studied by a variety of authors. The pioneering work on modeling Ginzburg-Landau vortices is the work of Bethuel, Brezis and Hélein [11]. In this work the authors suggested to consider a simplified Ginzburg-Landau model (1) with $a\equiv 1$ in Ω (*i.e.* without pinning term), in which the physical source of vortices, the external magnetic field, is modeled via a Dirichlet boundary condition with a positive degree on the boundary (3). The analysis of full Ginzburg-Landau functional, with induced and applied magnetic fields, was later performed by Sandier and Serfaty in [27].

The functional (1) with non-constant $a(x)$ was proposed by Rubinstein in [26] as a model of pinning vortices for Ginzburg-Landau minimizers. Shortly after, André and Shafrir [4] studied the asymptotics of minimizers for a smooth (say C^1) a. One of the first works to consider a discontinuous pinning term, which models a composite two-phase superconductor, was [18]. In this work, a single inclusion described by a pinning term independent of the parameter ε was considered for a simplified Ginzburg-Landau functional with Dirichlet boundary condition g on $\partial\Omega$. Namely the pinning term is

$$a(x) = \begin{cases} 1 & \text{if } x\in\Omega\setminus\omega\\ b & \text{if } x\in\omega\end{cases},$$

here ω is a simply connected open set s.t. $\overline{\omega}\subset\Omega$. The main objective of [18] was to establish that the vortices are attracted (pinned) by the inclusion ω, and their location inside ω can be obtained via minimization of certain finite-dimensional functional of renormalized energy. Full Ginzburg-Landau model with discontinuous pinning term was later considered by Aydi and Kachmar [5]. An ε-dependent but continuous pinning term $a_\varepsilon(x)$ was studied by Aftalion, Sandier and Serfaty in [1]. The work [3] studies the case of the smooth a with finite number of isolated zeros, and in [2] the pinning term a takes negative values in some regions of the domain Ω. The other works related to Ginzburg-Landau functional with pinning term include, *e.g.*, [21], [28].

In this work, we consider the minimization problem (1)-(3) with a discontinuous pinning term given by (2). We prove that despite the fact that $a_\varepsilon \to 1$ a.e. as $\varepsilon \to 0$, *i.e.* the pinning term disappears in the limit, the pinning domains ω_δ capture the vortices of Ginzburg-Landau minimizers of (1) for small ε.

The main difficulty in the analysis of this problem stems in the fact that the *a priori* Pohozhaev type estimate $\|1-|v|^2\|^2_{L^2(\Omega)} \le C\varepsilon^2$ for the minimizer v (on which the analysis in [11] and [18] is based) does not hold. Therefore, we develop a different strategy of reducing the study of the minimizers of (1) to the analysis of $\mathbb{S}^1$-valued maps via the uniform estimates on the modulus of minimizers away from the pinning domains (see Proposition 5 below).

Following [18], let U_ε be **the** unique global minimizer of E_ε in H^1 with $U_\varepsilon \equiv 1$ on $\partial\Omega$. This U_ε satisfies $b \le U_\varepsilon \le 1$. For $v \in H^1_g$ we define

$$F_\varepsilon(v) = F_\varepsilon(v,\Omega) := \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2|\nabla v|^2 + \frac{1}{2\varepsilon^2}U_\varepsilon^4(1-|v|^2)^2 \right\} \mathrm{d}x.$$

Using the Substitution Lemma of [18], we have that for $v \in H^1_g$,

$$E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v). \tag{4}$$

From the decomposition (4), we can reduce the minimization problem (1)-(3) to the minimization problem for F_ε in H^1_g, namely, the minimizer v_ε of F_ε in H^1_g has the same vorticity structure as the original minimizer u_ε of (1)-(3).

Depending on the relation between M (number of inclusions), and d (number of vortices), we distinguish two cases:
Case I: $M \ge d$ (more inclusions than vortices),
Case II: $M < d$ (more vortices than inclusions).

For example, we are going to show that for the minimizer v_ε:

- if $M = 3$ and $d = 2$ (Case I), we have two distinct inclusions containing exactly one zero each,
- if $M = 2$ and $d = 3$ (Case II), we have one zero inside one inclusion and two distinct zeros inside the other inclusion.

Generally speaking, outside a fixed neighborhood of $d' = \min\{d, M\}$ inclusions (centered at $\boldsymbol{a} = (a_{i_1}, ..., a_{i_{d'}})$), the minimizer v_ε is almost an $\mathbb{S}^1$-valued map. Moreover, by minimality of v_ε, the selection of centers of inclusion containing its zeros and the distribution of degrees of v_ε around the a_i's are related to the minimization of the Bethuel-Brezis-Hélein renormalized energy W_g. In other words, we reduce the problem of finding vortices of the minimizers v_ε to a two-step procedure. As the first step, we determine the inclusions with vortices, which is a discrete minimization problem for W_g and is significantly simpler then the minimization of this renormalized energy functional over $\Omega^{d'}$. Secondly, we determine the locations of the zeros (vortices) locally inside each inclusion and show that their positions depend only on b, on the geometry of ω and on the relation between d and M, but not on the external Dirichlet boundary condition g (see Theorem 4 below).

Our main result in Case I is the following:

Theorem 1. *Assume that $M \ge d$. Let v_ε be a minimizer of F_ε in $H^1_g(\Omega)$. For any sequence $\varepsilon_n \downarrow 0$, possibly after passing to a subsequence, there are d distinct points $\{a_{i_1}, ..., a_{i_d}\} \subset \{a_i, 1 \le i \le M\}$ and a function $v^* \in H^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{a_{i_1}, ..., a_{i_d}\}, \mathbb{S}^1)$ such that:*

1. *v^* is a harmonic map,* i.e.

$$\begin{cases} -\Delta v^* = v^*|\nabla v^*|^2 & \text{in } \Omega \setminus \{a_{i_1}, ..., a_{i_d}\} \\ v^* = g & \text{on } \partial\Omega \end{cases}. \tag{5}$$

2. *We have* $v_{\varepsilon_n} \to v^*$ *strongly in* $H^1_{\text{loc}}(\overline{\Omega} \setminus \{a_{i_1}, ..., a_{i_d}\})$ *and* $v_{\varepsilon_n} \to v^*$ *in* $C^\infty_{\text{loc}}(\Omega \setminus \{a_1, ..., a_M\})$.

3. v_{ε_n} *has* d *distinct vortices* $x_1^n, ..., x_d^n$ *such that* x_m^n *is inside* $\omega_\delta^{i_m}$, $m = 1, ..., d$ *and for small fixed* ρ, $\deg_{\partial B(x_i^n, \rho)}(v_{\varepsilon_n}) = 1$.

4. *The following expansion holds*

$$F_\varepsilon(v_\varepsilon) = \pi d b^2 |\ln \varepsilon| + \pi(1 - b^2) d |\ln \delta| + W_g((a_{i_1}, 1), ..., (a_{i_d}, 1)) + \tilde{W} + o_\varepsilon(1). \tag{6}$$

Here $\tilde{W} > 0$ *is a local renormalized energy depending only on* d, b *and* ω. *Moreover, the* d*-subset* $\{a_{i_1}, ..., a_{i_d}\} \subset \{a_1, ..., a_M\}$ *minimizes the Bethuel-Brezis-Hélein renormalized energy* W_g *among the* d*-subsets of* $\{a_1, ..., a_M\}$.

Remark 1. Here, W_g denotes the renormalized energy given by Theorem I.7 in [11] (with the degrees equal to 1 and the boundary data g). Its definition is recalled in Section 5.3.

The main result in Case II is

Theorem 2. *Assume that* $M < d$. *Let* v_ε *be a minimizer of* F_ε *in* $H^1_g(\Omega)$. *For any sequence* $\varepsilon_n \downarrow 0$, *possibly after passing to a subsequence, there is* $v^* \in H^1_{\text{loc}}(\overline{\Omega} \setminus \{a_1, ..., a_M\}, \mathbb{S}^1)$ *which satisfies* (5) *in* $\Omega \setminus \{a_1, ..., a_M\}$, *such that:*

1. $v_{\varepsilon_n} \to v^*$ *strongly in* $H^1_{\text{loc}}(\overline{\Omega} \setminus \{a_1, ..., a_M\})$ *and* $v_{\varepsilon_n} \to v^*$ *in* $C^\infty_{\text{loc}}(\Omega \setminus \{a_1, ..., a_M\})$.

2. *For* $\rho > 0$ *small,* v_{ε_n} *has exactly* $d_i := \deg_{\partial B(a_i, \rho)}(v_{\varepsilon_n})$ *zeros in* $B(a_i, \rho)$. *They are isolated, lie inside* ω_δ^i *and they have a degree equal to* 1.

3. $$\left[\frac{d}{M}\right] \le d_i \le \left[\frac{d}{M}\right] + 1, \text{ where } \left[\frac{d}{M}\right] \text{ is the integer part of } \frac{d}{M}. \tag{7}$$

Moreover, if $\frac{d}{M} = m_0 \in \mathbb{N}$, *then* $d_i \equiv m_0$, $1 \le i \le M$. *Otherwise, the configuration* $\{(a_1, d_1), ..., (a_M, d_M)\}$ *minimizes the renormalized energy* W_g *among the configurations* $\{(a_1, \tilde{d}_1), ..., (a_M, \tilde{d}_M)\}$. *Here* $\{a_i \,|\, 1 \le i \le M\}$ *are fixed and* $\tilde{d}_i \in \mathbb{Z}$ *are the subjects to the constraints (7) and* $\sum_{i=1}^M \tilde{d}_i = d$.

4. *The following expansion holds when* $\varepsilon \to 0$

$$\inf_{H^1_g} F_\varepsilon = \pi d b^2 |\ln \varepsilon| + \pi(\sum_{i=1}^M d_i^2 - d b^2) |\ln \delta| + W_g(\{\boldsymbol{a}, \boldsymbol{d}\}) + \tilde{W} + o_\varepsilon(1). \tag{8}$$

Here, $\{\boldsymbol{a}, \boldsymbol{d}\} = \{(a_1, d_1), ..., (a_M, d_M)\}$ *is a configuration given by the previous assertion and* $\tilde{W}$ *is local renormalized energy which depends only on* ω, b, d *and* M.

In both cases, we prove that the asymptotic location of the vortices inside a pinning domain depends only on b, ω and on the number of zeros inside the inclusion (see Theorem 4): this location is independent of the boundary data g on $\partial\Omega$.

2 Main tools

In this section we establish:

- Estimates for U_ε,
- Upper bounds for the energy of minimizers in Case I and Case II,
- An η-ellipticity estimate for minimizers.

2.1 Properties of U_ε

Proposition 1 (Maximum principle for U_ε, [18] Proposition 1)**.** *The special solution U_ε satisfies $b \le U_\varepsilon \le 1$ in Ω.*

Proposition 2. *There are $C, c > 0$ (independent of ε) s.t. for any $R > 0$ we have*

$$|a_\varepsilon - U_\varepsilon| \le C\mathrm{e}^{-\frac{cR}{\varepsilon}} \text{ in } V_R := \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\omega_\delta) \ge R\}, \tag{9}$$

$$|\nabla U_\varepsilon| \le \frac{C\mathrm{e}^{-\frac{cR}{\varepsilon}}}{\varepsilon} \text{ in } V_R. \tag{10}$$

The proof of the Proposition 2 is presented in the Appendix A.

2.2 Upper Bounds

Proposition 3. *Let $\xi = \dfrac{\varepsilon}{\delta}$.*

1. *Upper bound in Case I: $M \ge d$*

   There is a constant C depending only on g, ω and Ω s.t. we have

$$\inf_{H^1_g(\Omega)} F_\varepsilon(\cdot, \Omega) \le \pi d b^2 |\ln \xi| + \pi d |\ln \delta| + C. \tag{11}$$

2. *Upper bound in Case II: $M < d$*

   There is a constant C depending only on g, ω and Ω s.t. for all $d_1, ..., d_M \in \mathbb{N}$ s.t. $\sum d_i = d$ we have

$$\inf_{H^1_g(\Omega)} F_\varepsilon(\cdot, \Omega) \le \pi d b^2 |\ln \xi| + \pi \sum_i d_i^2 |\ln \delta| + C. \tag{12}$$

The proof of Proposition 3 is given in Appendix B.

2.3 Identifying bad discs

Lemma 1. *Let $g_\varepsilon, g_0 \in C^\infty(\partial\Omega, \mathbb{C})$ be s.t. $0 \le 1 - |g_\varepsilon| \le \varepsilon$ and $g_\varepsilon \to g_0$ in $C^1(\partial\Omega)$. Let also $\alpha_\varepsilon, \beta_\varepsilon \in L^\infty(\Omega, [b, 1])$.*

Consider the weighted Ginzburg-Landau functional

$$F^w_\varepsilon(v) = \frac{1}{2}\int_\Omega \left\{ \alpha_\varepsilon |\nabla v|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}(1 - |v|^2)^2 \right\}.$$

Denote v_ε a minimizer of F^w_ε in $H^1_{g_\varepsilon}$. Then the following results hold:

1. *Let $\chi = \chi_\varepsilon \in (0, 1)$ be s.t. $\chi \to 0$. There are $\varepsilon_0 > 0$, $C > 0$ and $C_1 > 0$ depending only on $b, \chi, \Omega, \|g_0\|_{C^1(\partial\Omega)}$ s.t for $\varepsilon < \varepsilon_0$, if*

$$F^w_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4}) \cap \Omega) \le \chi^2 |\ln \varepsilon| - C_1,$$

   then

$$|v_\varepsilon| \ge 1 - C\chi \text{ in } B(x, \varepsilon^{1/2}) \cap \Omega.$$

2. *Let $\mu \in (0, 1)$. Then there are $\varepsilon_0, C > 0$ depending only on $b, \mu, \Omega, \|g_0\|_{C^1(\partial\Omega)}$ s.t. for $\varepsilon < \varepsilon_0$, if*

$$F^w_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4}) \cap \Omega) \le C |\ln \varepsilon|,$$

   then

$$|v_\varepsilon| \ge \mu \text{ in } B(x, \varepsilon^{1/2}) \cap \Omega.$$

Lemma 1 is proved in Appendix C.

3 A model problem: one inclusion

By combining the results of Section 2, the proofs of both Theorem 1 and Theorem 2 are based on the analysis of two distinct problems:

1. A minimization problem of the Dirichlet functional among $\mathbb{S}^1$-valued map defined on a perforated domain.

2. The study of the minimizers v_ε around an inclusion.

This section focuses on the second problem. More precisely, we fix $\rho > 0$ and study the minimization problem of $F_\varepsilon(\cdot, B(a_i, \rho))$ with variable boundary conditions.

Fix $\rho > 0$ and let f_ε, $f_0 \in C^\infty(\partial B(0,\rho))$ be s.t. f_0 is $\mathbb{S}^1$-valued and s.t.

$$\|f_\varepsilon - f_0\|_{C^1(\partial B(0,\rho))} \to 0 \tag{13}$$

and

$$\||f_\varepsilon| - 1\|_{L^2(\partial B(0,\rho))} \leq C\varepsilon^2. \tag{14}$$

Assume also that $\deg_{\partial B(0,\rho)}(f_\varepsilon) = \deg_{\partial B(0,\rho)}(f_0) = d_0 > 0$.

For $i \in \{1, ..., M\}$ consider the minimization problem

$$F_\varepsilon(v, B(a_i, \rho)) := \frac{1}{2} \int_{B(a_i,\rho)} \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{1}{2\varepsilon^2} U_\varepsilon^4 (1 - |v|^2)^2 \right\} \mathrm{d}x \tag{15}$$

in the class

$$H^1_{f_\varepsilon,i} := \{v \in H^1(B(a_i, \rho), \mathbb{C}) \,|\, \mathrm{tr}_{\partial B(a_i,\rho)} v(x) = f_\varepsilon(x - a_i)\}. \tag{16}$$

Without loss of generality assume $a_i = 0$. Let v_ε be a minimizer of (15) in (16). Performing the change of variables $\hat{x} = \frac{x}{\delta}$ in (15), we have

$$F_\varepsilon(v_\varepsilon, B(0, \rho)) = \hat{F}_\xi(\hat{v}_\varepsilon, B(0, \frac{\rho}{\delta})) := \frac{1}{2} \int_{B(0,\frac{\rho}{\delta})} \left\{ \hat{U}_\varepsilon^2 |\nabla \hat{v}|^2 + \frac{1}{2\xi^2} \hat{U}_\varepsilon^4 (1 - |\hat{v}|^2)^2 \right\} \mathrm{d}\hat{x}. \tag{17}$$

Here, for a map $w \in H^1(B(0, \rho))$, we denote $\hat{w}(\hat{x}) := w(\delta \hat{x})$ and $\xi = \dfrac{\varepsilon}{\delta}$. The class (16) under this change of variables becomes

$$\hat{H}^1_{f_\varepsilon} := \left\{ \hat{v} \in H^1(B(0, \frac{\rho}{\delta}), \mathbb{C}) \,|\, \mathrm{tr}_{\partial B(0,\frac{\rho}{\delta})} \hat{v}(\cdot) = f_\varepsilon(\delta \cdot) \right\}. \tag{18}$$

Note that the above rescaling enables us to fix the pinning domain independently of ε.

The asymptotic behavior of $\hat{v}_\varepsilon$ will be obtained in several steps:

- We first establish a bound for $|\hat{v}_\varepsilon|$ (Proposition 5). This bound will allow us to localize (roughly) the vortices of v_ε near the inclusion.

- We next establish sharp energy estimates (Proposition 6) and use them to obtain the uniform convergence of solutions away from the inclusion (Proposition 7 and Corollary 2). We establish the strong H^1 convergence of solutions away from the "vortices" (Proposition 8) and derive the equation satisfied by the limiting map (Proposition 10).

- The last step is the location and quantization of the vorticity: for small ε, the minimizers admits exactly d_0 zeros, and all the zeros lie in the inclusion and have a degree equal to 1 (Propositions 8 and 11).

Following the same lines as for Proposition 3, one may prove

Proposition 4. *Let $\hat{v}_\varepsilon$ be a minimizer of $\hat{F}_\xi$ in (16). Then there is a constant C independent of ε s.t. we have*

$$F_\varepsilon(v_\varepsilon, B(0, \rho)) = \hat{F}_\xi(\hat{v}_\varepsilon, B(0, \frac{\rho}{\delta})) \leq \pi d_0 b^2 |\ln \xi| + \pi d_0^2 |\ln \delta| + C. \tag{19}$$

3.1 Uniform convergence of $|\hat{v}_\varepsilon|$ to 1 away from inclusions

Proposition 5. *Let $K \subset \mathbb{R}^2$ be a compact set such that $\omega \subset K$ and $\operatorname{dist}(\partial K, \omega) > 0$. Then there is $C > 0$ independent of ε s.t. for sufficiently small ε we have*

$$|\hat{v}_\varepsilon| \geq 1 - C|\ln\varepsilon|^{-1/3} \text{ in } B_{\frac{\rho}{\delta}} \setminus K.$$

Proof. Using Lemma 1 with $\chi = |\ln\varepsilon|^{-1/3}$, we find that there exist $C, C_1 > 0$ s.t. for $\varepsilon > 0$ small, if $F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4})) < |\ln\varepsilon|^{\frac{1}{3}} - C_1$ then $|v_\varepsilon| \geq 1 - C\chi$ in $B(x, \varepsilon^{1/2})$.

We argue by contradiction. Assume that there exists a compact K containing ω s.t. $\operatorname{dist}(\partial K, \omega) > 0$ and s.t., up to a subsequence, there is a sequence of points $\hat{x}_\varepsilon \in B(0, \frac{\rho}{\delta}) \setminus K$ s.t. $|\hat{v}_\varepsilon(\hat{x}_\varepsilon)| < 1 - C|\ln\varepsilon|^{-1/3}$ with C given by Lemma 1. Note that $\hat{x}_\varepsilon \in B(0, \frac{\rho}{\delta}) \setminus K$ corresponds to $x_\varepsilon \in B(0,\rho) \setminus (\delta \cdot K)$. From Lemma 1 and Proposition 2

$$\frac{1}{2}\int_{B(x_\varepsilon, \varepsilon^{1/4})} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \geq |\ln\varepsilon|^{1/3} - \mathcal{O}(1). \tag{20}$$

We claim that due to the conditions (13), (14), we may extend v_ε (keeping the same notation for the extension) to a smooth map, still denoted v_ε, s.t.

$$\begin{cases} v_\varepsilon(x) = x^{d_0}/|x|^{d_0} \text{ in } B(0,3\rho) \setminus \overline{B(0,2\rho)} \\ \displaystyle\int_{B(0,3\rho)\setminus\overline{B(0,\rho)}} (1 - |v_\varepsilon|^2)^2 \leq C\varepsilon^2 \\ |\nabla v_\varepsilon| \leq C \text{ with } C > 0 \text{ is independent of } \varepsilon \end{cases}. \tag{21}$$

To make the above extension explicit, choose $\zeta \in C^\infty(\mathbb{R}^+, [0,1])$ s.t. $\zeta \equiv 0$ in $[0,\rho]$ and $\zeta \equiv 1$ in $[2\rho, 3\rho]$ and take

$$v_\varepsilon(s e^{\imath\theta}) = \Big[\zeta(s) + (1 - \zeta(s))|f_\varepsilon(\rho e^{\imath\theta})|\Big]\, e^{\imath\left[d_0\theta + (1-\zeta(s))\phi_\varepsilon(\rho e^{\imath\theta})\right]}.$$

Here $x = s e^{\imath\theta}$, $s > 0$ and $\phi_\varepsilon \in C^\infty(\partial B(0,\rho), \mathbb{R})$ is s.t. $f_\varepsilon(\rho e^{\imath\theta}) = |f_\varepsilon| e^{\imath(d_0\theta + \phi_\varepsilon)}$. Consequently, as follows from (11) and (21), this map satisfies

$$\frac{1}{2}\int_{B(0,3\rho)} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \leq C|\ln\varepsilon|.$$

Therefore, the map v_ε in $B(0,3\rho)$ fulfills the conditions of Theorem 4.1 in [27]. This theorem guarantees that:

- we may cover the set $\{x \in B(0, 3\rho - \varepsilon/b) \,|\, |v_\varepsilon(x)| < 1 - (\varepsilon/b)^{1/8}\}$ with a finite collection of disjoint balls $\mathcal{B}^\varepsilon := \{B_j^\varepsilon\}$;
- the radius of $\mathcal{B}^\varepsilon$, $\operatorname{rad}(\mathcal{B}^\varepsilon)$, which is defined as the sum of the radii of the balls B_j^ε, $\operatorname{rad}(\mathcal{B}^\varepsilon) := \sum_j \operatorname{rad}(B_j^\varepsilon)$, satisfies $\operatorname{rad}(\mathcal{B}^\varepsilon) \leq 10^{-2}\delta \cdot \operatorname{dist}(\omega, \partial K)$;
- denoting $d_j = \deg_{\partial B_j^\varepsilon}(v_\varepsilon)$ if $B_j^\varepsilon \subset B(0, 3\rho - \varepsilon/b)$ and $d_j = 0$ otherwise;

we have

$$\frac{1}{2}\int_{\mathcal{B}^\varepsilon} \left\{ |\nabla v_\varepsilon|^2 + \frac{b^2}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \geq \pi \sum_j |d_j| \ln\frac{\delta}{\varepsilon} - C. \tag{22}$$

Note that, by the construction of v_ε in $B(0,3\rho) \setminus \overline{B(0,\rho)}$, if we have $\deg_{\partial B_j^\varepsilon}(v_\varepsilon) \neq 0$ then $B_j^\varepsilon \subset B(0, 5\rho/2)$. Thus $d_j = \deg_{\partial B_j^\varepsilon}(v_\varepsilon)$ for all j.

In order to obtain a lower bound for F_ε we use the identity

$$F_\varepsilon(v_\varepsilon, B(x_\varepsilon,\varepsilon^{1/4})\cup\mathcal{B}^\varepsilon) = \frac{b^2}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})\cup\mathcal{B}^\varepsilon}\left\{|\nabla v_\varepsilon|^2+\frac{b^2}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} + \frac{1}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})\cup\mathcal{B}^\varepsilon}\Big\{(U_\varepsilon^2-b^2)|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(U_\varepsilon^4-b^4)(1-|v_\varepsilon|^2)^2\Big\}. \tag{23}$$

The first integral in (23) is estimate via (22):

$$\begin{aligned}\frac{b^2}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})\cup\mathcal{B}^\varepsilon}\left\{|\nabla v_\varepsilon|^2+\frac{b^2}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} &\geq \pi b^2\sum_j|\deg_{\partial B_j}(v_\varepsilon)|\ln\frac{\delta}{\varepsilon}-C\\ &\geq \pi b^2 d_0\ln\frac{\delta}{\varepsilon}-C_0. \end{aligned}\tag{24}$$

By combining (20) and Proposition 2, we have for small ε

$$\begin{aligned}&\frac{1}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})\cup\mathcal{B}^\varepsilon}\left\{(U_\varepsilon^2-b^2)|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}(U_\varepsilon^4-b^4)(1-|v_\varepsilon|^2)^2\right\}\\ &\geq \frac{1-b^2}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})}\left\{|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\}-C\geq(1-b^2)|\ln\varepsilon|^{1/3}-C';\end{aligned}\tag{25}$$

here we rely on the assumption (H) on the behavior of $\delta(\varepsilon)$ as $\varepsilon\to 0$.

Substituting the bounds (24) and (25) in (23) we obtain a contradiction with (11). This completes the proof of Proposition 5. □

3.2 Distribution of Energy in $B(0,\frac{\rho}{\delta})$

Proposition 6. *The following estimates hold:*

$$\frac{1}{2}\int_{B(0,\rho/\delta)\setminus\overline{B(0,1)}}\hat{U}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2=\pi d_0^2|\ln\delta|+\mathcal{O}(1), \tag{26}$$

and (recall that $\xi=\dfrac{\varepsilon}{\delta}$*)*

$$\hat{F}_\xi(\hat{v}_\varepsilon,B(0,1))=\pi d_0 b^2|\ln\xi|+\mathcal{O}(1). \tag{27}$$

Proof. We start by proving that

$$\hat{F}_\xi(\hat{v}_\varepsilon,B(0,1))\geq\pi d_0 b^2|\ln\xi|-\mathcal{O}(1). \tag{28}$$

As before, we use Theorem 4.1 in [27]: for $0<r<r_0:=10^{-2}\cdot\mathrm{dist}(\omega,\partial B(0,1))$, there are $C>0$ and a finite covering by disjoint balls $B_1^\varepsilon,...,B_N^\varepsilon$ (with the sum of radii at most r) of the set $\{\hat{x}\in B(0,1-\xi/b)\,|\,1-|\hat{v}_\varepsilon(\hat{x})|\geq(\xi/b)^{1/8}\}$ s.t.

$$\frac{1}{2}\int_{\cup_j B_j^\varepsilon}\left\{|\nabla\hat{v}_\varepsilon|^2+\frac{b^2}{2\xi^2}(1-|\hat{v}_\varepsilon|^2)^2\right\}\geq\pi D|\ln\xi|-C, \tag{29}$$

with $D=\sum_j|d_j|$ and

$$d_j=\begin{cases}\deg_{\partial B_j^\varepsilon}(\hat{v}_\varepsilon) & \text{if } B_j^\varepsilon\subset B(0,1-\xi/b)\\ 0 & \text{otherwise}\end{cases}.$$

From Proposition 5, for ε small, if $\deg_{\partial B_j^\varepsilon}(\hat{v}_\varepsilon) \neq 0$ then $B_j^\varepsilon \subset B(0,1-r_0) \subset B(0,1-\xi/b)$. It follows that $D \geq d_0$ and then (28) is a direct consequence of (29) and the bound $\hat{U}_\varepsilon \geq b$.

We next prove that there is $C > 0$ s.t.

$$\frac{1}{2}\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} \hat{U}_\varepsilon^2|\nabla \hat{v}_\varepsilon|^2 \geq \pi d_0^2|\ln\delta| - C. \tag{30}$$

By Proposition 5, $|v_\varepsilon| \geq 1/2$ in $B(0,\frac{\rho}{\delta}) \setminus \overline{B(0,1)}$, therefore, $\hat{w}_\varepsilon := \frac{\hat{v}_\varepsilon}{|\hat{v}_\varepsilon|}$ is well-defined in this domain. Observe that

$$\frac{1}{2}\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} |\nabla \hat{w}_\varepsilon|^2 \geq \frac{1}{2}\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} \left|\nabla \frac{z^{d_0}}{|z|^{d_0}}\right|^2 = \pi d_0^2 \ln\frac{\rho}{\delta}. \tag{31}$$

We claim that (30) holds with $C = \pi d_0^2|\ln\rho| + 1$ (for small ε). By contradiction, assume (30) does not hold. Then, up to a subsequence, we have

$$\frac{1}{2}\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} \hat{U}_\varepsilon^2|\nabla \hat{v}_\varepsilon|^2 < \pi d_0^2\ln\frac{\rho}{\delta} - 1. \tag{32}$$

On the other hand, we have

$$|\nabla \hat{v}_\varepsilon|^2 = |\hat{v}_\varepsilon|^2|\nabla \hat{w}_\varepsilon|^2 + |\nabla|\hat{v}_\varepsilon||^2$$

and therefore

$$\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} |\nabla \hat{v}_\varepsilon|^2 \geq \int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} |\nabla \hat{w}_\varepsilon|^2 - \int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} (1-|\hat{v}_\varepsilon|^2)|\nabla \hat{w}_\varepsilon|^2. \tag{33}$$

Since $|\hat{v}_\varepsilon| \geq \frac{1}{2}$ in $B(0,\frac{\rho}{\delta}) \setminus \overline{B(0,1)}$ we have $|\nabla \hat{w}_\varepsilon| \leq 2|\nabla \hat{v}_\varepsilon|$. Therefore, by (32), Proposition 5 and (H) we estimate the last term in (33):

$$\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} (1-|\hat{v}_\varepsilon|^2)|\nabla \hat{w}_\varepsilon|^2 \leq C_2|\ln\varepsilon|^{-\frac{1}{3}}\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} |\nabla \hat{v}_\varepsilon|^2 \leq C_3\frac{|\ln\delta|}{|\ln\varepsilon|^{\frac{1}{3}}} \to 0. \tag{34}$$

Combining (31), (33) and (34), we find that

$$\int_{B(0,\frac{\rho}{\delta})\setminus\overline{B(0,1)}} |\nabla \hat{v}_\varepsilon|^2 \geq \pi d_0^2\ln\frac{\rho}{\delta} - o_\varepsilon(1).$$

Since $|\hat{U}_\varepsilon - 1| \leq C\xi^4$ in $B_{\frac{\rho}{\delta}} \setminus \overline{B(0,1)}$ (see Proposition 2), we obtain a contradiction with (32), and (30) follows. Comparing the lower bounds (28) and (30) with the upper bound in Proposition 4, the Proposition 6 follows. □

Using exactly the same techniques as in the proof of Proposition 6, one may easily prove the following estimate.

Corollary 1. *For any $R_2 > R_1 \geq 1$*

$$F_\xi(\hat{v}_\varepsilon, B(0,R_2) \setminus \overline{B(0,R_1)}) = \mathcal{O}(1).$$

3.3 Convergence in $C^\infty(K)$ for a compact K s.t. $K \cap \overline{\omega} = \emptyset$

Proposition 7. *Let $K \subset \mathbb{R}^2 \setminus \overline{\omega}$ be a smooth compact set. Then we have*

$$\hat{v}_\varepsilon \text{ is bounded in } C^k(K) \text{ for all } k \geq 0 \tag{35}$$

and there is $C_K > 0$ s.t.

$$|\hat{v}_\varepsilon| \geq 1 - C_K\xi^2 \text{ in } K. \tag{36}$$

Proof. From Proposition 2

$$E_\xi(\hat{U}_\varepsilon, K) = \frac{1}{2}\int_K |\nabla \hat{U}_\varepsilon|^2 + \frac{1}{2\xi^2}(1-\hat{U}_\varepsilon^2)^2 = \mathcal{O}(1). \tag{37}$$

As in [18], the following expansion holds

$$E_\xi(\hat{U}_\varepsilon \hat{v}_\varepsilon, K) = E_\xi(\hat{U}_\varepsilon, K) + \hat{F}_\xi(\hat{v}_\varepsilon, K) + \int_{\partial K} (|\hat{v}_\varepsilon|^2 - 1)\hat{U}_\varepsilon \partial_\nu \hat{U}_\varepsilon. \tag{38}$$

Using (10), we have

$$\int_{\partial K} (|\hat{v}_\varepsilon|^2 - 1)\hat{U}_\varepsilon \partial_\nu \hat{U}_\varepsilon = o_\varepsilon(1).$$

With (37) and (38), we conclude that $E_\xi(\hat{U}_\varepsilon \hat{v}_\varepsilon, K) = \mathcal{O}(1)$. Since $\hat{U}_\varepsilon$ and $\hat{U}_\varepsilon \hat{v}_\varepsilon$ satisfy the Ginzburg-Landau equation $-\Delta u = \frac{1}{\xi^2} u(1-|u|^2)$ in K, as well as $|\hat{U}_\varepsilon| \leq 1$ and $|\hat{U}_\varepsilon \hat{v}_\varepsilon| \leq 1$. Theorem 1 in [23] implies that

$$\hat{U}_\varepsilon \text{ and } \hat{U}_\varepsilon \hat{v}_\varepsilon \text{ are bounded in } C^k(K) \text{ for all } k \geq 0.$$

It follows that $\hat{v}_\varepsilon$ is bounded in $C^k(K)$ for each $k \geq 0$. On the other hand, using the fact that $\hat{v}_\varepsilon$ is bounded in $C^k(K)$ together with the equation of $\hat{v}_\varepsilon$, we find that $1 - |\hat{v}_\varepsilon|^2 \leq C_K \xi^2$ in K. □

Corollary 2. *For $K \subset \mathbb{R}^2 \setminus \overline{\omega}$, up to a subsequence, there is some $v_0 \in C^\infty(K, \mathbb{S}^1)$ s.t. $\hat{v}_\varepsilon \to v_0$ in $C^\infty(K)$.*

We are now in position to bound the potential part of the energy.

Corollary 3. *There exists $C > 0$ independent of ε s.t.*

$$\frac{1}{\varepsilon^2}\int_{B(0,\rho)} (1-|v_\varepsilon|^2)^2 = \frac{1}{\xi^2}\int_{B(0,\rho/\delta)} (1-|\hat{v}_\varepsilon|^2)^2 \leq C. \tag{39}$$

Proof. Note that from Propositions 4 and 6, we find that there is $C > 0$ s.t.

$$\frac{1}{\xi^2}\int_{B(0,\rho/\delta)\setminus\overline{B(0,1)}} (1-|\hat{v}_\varepsilon|^2)^2 \leq C.$$

Thus it remains to prove the estimate in $B(0,1)$ for small ε. Using (35), $\mathrm{tr}_{\partial B(0,1)}\hat{v}_\varepsilon$ is bounded in $C^1(\partial B(0,1))$ and $1 - |\hat{v}_\varepsilon|^2 \leq C\xi^2$ on $\partial B(0,1)$ (for small ε). These properties, allow us to construct a smooth extension $\tilde{v}_\varepsilon$ of $\mathrm{tr}_{\partial B(0,1)}\hat{v}_\varepsilon$ into $B(0,2) \setminus \overline{B(0,1)}$, s.t. $h = \mathrm{tr}_{\partial B(0,2)}\tilde{v}_\varepsilon$ is $\mathbb{S}^1$-valued and independent of ε, $1 - |\tilde{v}_\varepsilon|^2 \leq C\xi^2$ in $B(0,2) \setminus \overline{B(0,1)}$ and

$$\int_{B(0,2)\setminus\overline{B(0,1)}} \left\{ |\nabla \tilde{v}_\varepsilon|^2 + \frac{1}{2\xi^2}(1-|\tilde{v}_\varepsilon|^2)^2 \right\} \leq C_0. \tag{40}$$

(For example, this construction is performed by mimicking (21))

Define w_ε as $w_\varepsilon = \hat{v}_\varepsilon$ in $B(0,1)$ and $w_\varepsilon = \tilde{v}_\varepsilon$ in $B(0,2) \setminus \overline{B(0,1)}$. Clearly, $w_\varepsilon \in H^1_h(B(0,2))$, w_ε is bounded in $L^2(B(0,2))$ and, thanks to Proposition 6 and (40),

$$\frac{1}{2}\int_{B(0,2)} \left\{ |\nabla w_\varepsilon|^2 + \frac{b^2}{2\xi^2}(1-|w_\varepsilon|^2)^2 \right\} \leq \pi d_0 |\ln \xi| + C_0.$$

We may now apply Proposition 0.1 in [14] to w_ε in $B(0,2)$ to conclude that $\frac{1}{\xi^2}\int_{B(0,2)} (1-|w_\varepsilon|^2)^2 \leq C_1$. Therefore the bound (39) holds. □

3.4 The bad discs

Consider a family of discs $(B(x_i,\varepsilon^{1/4}))_{i\in I}$ such that (here I depends on ε)

$$\text{for all } i \in I \text{ we have } x_i \in \Omega,$$

$$B(x_i,\varepsilon^{1/4}/4)\cap B(x_i,\varepsilon^{1/4}/4)=\emptyset \text{ if } i\neq j,$$
$$\cup_{i\in I}B(x_i,\varepsilon^{1/4})\supset\Omega.$$

For $\mu\in(1/2,1)$, let $C=C(\mu)$, $\varepsilon_0=\varepsilon_0(\mu)$ be defined as in the second part of Lemma 1. For $\varepsilon<\varepsilon_0$, we say that $B(x_i,\varepsilon^{1/4})$ is *μ-good disc* if

$$F_\varepsilon(v_\varepsilon,B(x_i,\varepsilon^{1/4})\cap\Omega)\leq C(\mu)|\ln\varepsilon|$$

and $B(x_i,\varepsilon^{1/4})$ is *μ-bad disc* if

$$F_\varepsilon(v_\varepsilon,B(x_i,\varepsilon^{1/4})\cap\Omega)> C(\mu)|\ln\varepsilon|. \tag{41}$$

Let $J_\varepsilon=J:=\{i\in I\,|\,B(x_i,\varepsilon^{1/4})$ is a μ-bad disc$\}$.

Lemma 2. *There is an integer N, which depends only on g and μ, s.t.*

$$\text{Card}\,J\leq N.$$

Proof. Since each point of Ω is covered by at most 16 discs $B(x_i,\varepsilon^{1/4})$, we have

$$\sum_{i\in I}F_\varepsilon(v_\varepsilon,B(x_i,\varepsilon^{1/4})\cap\Omega)\leq 16F_\varepsilon(v_\varepsilon,\Omega).$$

The previous assertion implies that Card $J\leq\frac{16C_0}{C_1(\mu)}$. $\square$

The next result is a straightforward variant of Theorem IV.1 in [11].

Lemma 3. *Possibly after passing to a subsequence and relabeling I, we may choose $J'\subset J$ and a constant $\lambda\geq1$ (independently of ε) s.t.*

$$J'=\{1,...,N'\},\ N'=\text{Cst},$$

$$|x_i-x_j|\geq 8\lambda\varepsilon^{1/4}\ \textit{for } i,j\in J',\ i\neq j$$

and

$$\cup_{i\in J}B(x_i,\varepsilon^{1/4})\subset\cup_{i\in J'}B(x_i,\lambda\varepsilon^{1/4}).$$

We will say that, for $i\in J'$, $B(x_i,\lambda\varepsilon^{1/4})$ are *separated μ-bad discs.* From now on, we work with separated μ-bad discs. Denote $\hat{x}_i=\dfrac{x_i}{\delta}$. By Proposition 5 we know that for small ε, we have $\hat{x}_i\in\overline{B_1}$. Clearly, up to a subsequence,

$$\left\{\begin{array}{c}\text{there are }\alpha_1,...,\alpha_\kappa,\ \kappa\text{ distinct points in }\overline{B_1}\\ \{\Lambda_1,...,\Lambda_\kappa\}\text{ a partition (in non empty sets) of }J'\text{ s.t.}\\ \text{for }i\in J',\ \text{if }i\in\Lambda_k\text{ then }\hat{x}_i\to\alpha_k.\end{array}\right. \tag{42}$$

Note that for $i\in J'$, we have

$$y\in\{\alpha_1,...,\alpha_\kappa\}\iff\begin{cases}\forall\,\eta>0,\text{ for small }\varepsilon,\\ \text{there is a }\mu\text{-bad disc inside }B(y,\eta).\end{cases} \tag{43}$$

3.5 Convergence in $H^1_{\rm loc}(\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\})$

We have the following theorem.

Proposition 8. *Let $\alpha_1,...,\alpha_\kappa$ be defined by* (42)*. Then we have:*

1. *The points $\alpha_1,...,\alpha_\kappa$ belong to ω.*
2. *There exists $v_0\in H^1_{\rm loc}(\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\},\mathbb{S}^1)$ s.t. (possibly after extraction)*
$$\hat{v}_\varepsilon\to v_0 \text{ in } H^1_{\rm loc}(\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\})\tag{44}$$
$$\hat{v}_\varepsilon\to v_0 \text{ in } C^0_{\rm loc}(\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\}).\tag{45}$$
3. *There exists $\eta_0>0$ s.t. for all $0<\eta<\eta_0$ and for sufficiently small ε we have*
$$\deg_{\partial B_\eta(\alpha_k)}(\hat{v}_\varepsilon/|\hat{v}_\varepsilon|)=\deg_{\partial B_{\eta_0}(\alpha_k)}(v_0)=1.$$
4. *$\kappa=d_0$.*

Proof. **Step 1:** $\hat{v}_\varepsilon\rightharpoonup v_0$ in $H^1_{\rm loc}(\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\})$, $v_0\in H^1_{\rm loc}(\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\},\mathbb{S}^1)$ and $\alpha_k\in\omega$

Proposition 5 guarantees that $\alpha_1,...,\alpha_\kappa\in\overline{\omega}$. Let
$$\eta_0=\begin{cases}10^{-2}\cdot\min_{k\neq k'}|\alpha_k-\alpha_{k'}| & \text{if } \kappa>1\\ 1 & \text{if } \kappa=1\end{cases}.\tag{46}$$

Applying Theorem 4.1 in [27] we have for all $0<\eta<\eta_0$ and for small ε
$$\frac{1}{2}\int_{\cup_{k\in\{1,...,\kappa\}}B(\alpha_k,\eta)}\left\{|\nabla\hat{v}_\varepsilon|^2+\frac{b^2}{2\xi^2}(1-|\hat{v}_\varepsilon|^2)^2\right\}\geq\pi d_0\ln\frac{\eta}{\xi}-C\tag{47}$$
with C independent of ε and η. Combining (47) with (27) and Corollary 1 we obtain that $\hat{v}_\varepsilon$ is bounded in $H^1(K)$; here $K\subset\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\}$ is an arbitrary compact set. Therefore, there exists $v_0\in H^1_{\rm loc}(\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\})$ s.t. we have $\hat{v}_\varepsilon\rightharpoonup v_0$ in $H^1_{\rm loc}(\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\})$ (possibly passing to a subsequence). Since $\|1-|\hat{v}_\varepsilon|\|_{L^2(K)}\to 0$ for all compact sets $K\subset\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\}$, we find that v_0 is $\mathbb{S}^1$- valued.

Following the proof of Step 7 in Theorem C in [18], we can prove that $\alpha_1,...,\alpha_\kappa\notin\partial\omega$, thus $\alpha_1,...,\alpha_\kappa\in\omega$, and the first assertion follows.

Step 2: Proof of *2.*

Adapting the techniques of [10] (Theorem 2, Step 1), we establish (44) and (45) in a ball $B=B(y,R_0)$ s.t. $\overline{B}\subset\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\}$.

Let $y\in\mathbb{R}^2$ and let $R'>R>0$ be s.t. $B(y,R')\subset\mathbb{R}^2\setminus\{\alpha_1,...,\alpha_\kappa\}$. Since $\hat{F}_\xi(\hat{v}_\varepsilon,B(y,R'))$ is bounded independently on ε, there is $R_0\in(R,R')$ (independent of ε) s.t., passing to a further subsequence if necessary we have
$$\int_{\partial B(y,R_0)}\left\{|\partial_\tau\hat{v}_\varepsilon|^2+\frac{1}{\xi^2}(1-|\hat{v}_\varepsilon|^2)^2\right\}\leq C \text{ with } C \text{ independent of } \varepsilon.\tag{48}$$

Indeed, for $r\in(R,R')$ denote
$$I_\varepsilon(r)=\int_{\partial B(y,r)}\left\{|\nabla\hat{v}_\varepsilon|^2+\frac{1}{\xi^2}(1-|\hat{v}_\varepsilon|^2)^2\right\}.$$

Using the Fubini theorem and the Fatou Lemma we have

$$0 \le \int_R^{R'} \liminf_\varepsilon I_\varepsilon(r)\,\mathrm{d}r \le \liminf_\varepsilon \int_R^{R'} I_\varepsilon(r)\,\mathrm{d}r \le C'.$$

Consequently, $\liminf_\varepsilon I_\varepsilon(r) < \infty$ for almost all $r \in (R, R')$, so that (48) holds with $C = \dfrac{C'}{R' - R}$.

Let $g_\varepsilon = \mathrm{tr}_{\partial B}\hat{v}_\varepsilon$. Since $|\hat{v}_\varepsilon| \ge 1/2$ in $B = B(y, R_0)$, we have $\deg_{\partial B}(g_\varepsilon) = 0$. The bound (48) implies that, up to choose a subsequence, g_ε is weakly convergent in $H^1(\partial B)$. Consequently there is $h \in H^1(\partial B, \mathbb{S}^1)$, $h = \mathrm{e}^{\imath\varphi}$, $\varphi \in H^1(\partial B, \mathbb{R})$ s.t.

$$g_\varepsilon \to h \text{ uniformly on } \partial B, \tag{49}$$

$$g_\varepsilon \to h \text{ in } H^{1/2}(\partial B). \tag{50}$$

Let $\eta_\varepsilon : B \to \mathbb{R}^+$ be the minimizer of $\displaystyle\int_B \left\{ |\nabla\eta|^2 + \frac{1}{\xi^2}(1-\eta)^2 \right\}$ in $H^1_{|g_\varepsilon|}(B, \mathbb{R})$. Then η_ε satisfies

$$\begin{cases} -\xi^2\Delta\eta_\varepsilon + \eta_\varepsilon = 1 & \text{in } B \\ \eta_\varepsilon = |g_\varepsilon| & \text{on } \partial B \end{cases}.$$

It follows from [10] that

$$\int_B \left\{ |\nabla\eta_\varepsilon|^2 + \frac{1}{\xi^2}(1-\eta_\varepsilon)^2 \right\} \le C\xi. \tag{51}$$

Using (49), there is $\varphi_\varepsilon \in H^1(\partial B, \mathbb{R})$, s.t. $g_\varepsilon = |g_\varepsilon|\mathrm{e}^{\imath\varphi_\varepsilon}$ and $\varphi_\varepsilon \to \varphi$ uniformly on ∂B. Following [10], denote by $\psi_\varepsilon \in H^1_{\varphi_\varepsilon}(B, \mathbb{R})$ the unique solution of $-\mathrm{div}(a^2\nabla\psi_\varepsilon) = 0$. (Here $a = b$ in ω and $a = 1$ in $\mathbb{R}^2 \setminus \omega$.) From (50), $\psi_\varepsilon \to \psi$ in $H^1(B)$ where $\psi \in H^1_\varphi(B, \mathbb{R})$ is the unique solution of $-\mathrm{div}(a^2\nabla\psi) = 0$. Since $\eta_\varepsilon\mathrm{e}^{\imath\psi_\varepsilon} \in H^1_{g_\varepsilon}(B)$, we have

$$\hat{F}_\xi(\hat{v}_\varepsilon, B) \le \hat{F}_\xi(\eta_\varepsilon\mathrm{e}^{\imath\psi_\varepsilon}, B) \le \frac{1}{2}\int_B \hat{U}_\varepsilon^2|\nabla\psi_\varepsilon|^2 + C\xi \underset{\varepsilon\to 0}{\to} \frac{1}{2}\int_B a^2|\nabla\psi|^2. \tag{52}$$

On the other hand, since $\hat{v}_\varepsilon \rightharpoonup v_0$ in $H^1(B)$, we have $v_0 = \mathrm{e}^{\imath\phi}$ with $\phi \in H^1_\varphi(B, \mathbb{R})$ and

$$\liminf_\varepsilon \hat{F}_\xi(\hat{v}_\varepsilon, B) \ge \liminf_\varepsilon \frac{1}{2}\int_B \hat{U}_\varepsilon^2|\nabla\hat{v}_\varepsilon|^2 \ge \frac{1}{2}\int_B a^2|\nabla v_0|^2 = \frac{1}{2}\int_B a^2|\nabla\phi|^2. \tag{53}$$

(The last inequality follows from $\hat{U}_\varepsilon \to a$ in L^2, $|U_\varepsilon| \le 1$ and $\hat{v}_\varepsilon \rightharpoonup v_0$ in H^1.)

By combining (52), (53) and the fact that ψ minimizes $\displaystyle\int_B a^2|\nabla\cdot|^2$ in $H^1_\varphi(B, \mathbb{R})$, we find that (44) holds. Furthermore, the map ψ in (52) is the same as ϕ in (53).

Note that since

$$\frac{1}{2}\int_B \hat{U}_\varepsilon^2 \left|\nabla\frac{\hat{v}_\varepsilon}{|\hat{v}_\varepsilon|}\right|^2 - o_\varepsilon(1) \le \hat{F}_\xi(\hat{v}_\varepsilon, B),$$

by comparing (52) with (53), we also have

$$\int_K |\nabla|\hat{v}_\varepsilon||^2 + \frac{1}{\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2 \to 0. \tag{54}$$

In order to prove (45), it suffices to establish the convergence

$$\phi_\varepsilon \to \phi \text{ in } L^\infty(B) \text{ with } \phi_\varepsilon \in H^1_{\varphi_\varepsilon}(B, \mathbb{R}) \text{ and } \hat{v}_\varepsilon = |\hat{v}_\varepsilon|\mathrm{e}^{\imath\phi_\varepsilon}, \tag{55}$$

and to use the fact that $|\hat{v}_\varepsilon| \to 1$ uniformly.

Proof of (55). If $\partial\omega \cap B = \emptyset$, then the argument is the same as in [10]. Assume next that $\partial\omega \cap B \neq \emptyset$, and let $\tilde{\psi} \in H^{3/2}(B,\mathbb{R})$ be the harmonic extension of φ. Since $\zeta := \phi - \tilde{\psi} \in H^1_0(B,\mathbb{R})$ satisfies $-\mathrm{div}(a^2\nabla\zeta) = \mathrm{div}(a^2\nabla\tilde{\psi})$, Theorem 1 in [22] implies that $\phi \in W^{1,p}(B,\mathbb{R})$ for some $p > 2$.

We next prove that, for some $q > 2$ and $\tilde{B} = B(y',\tilde{R})$ s.t. $B(y',2\tilde{R}) \subset B$, we have $\|\phi_\varepsilon - \phi\|_{W^{1,q}(\tilde{B})} \to 0$. (Once proved, this assertion will imply, via Sobolev embedding that (55) holds.)

Note that (up to a subsequence) $\phi_\varepsilon \to \phi$ in $L^2(B,\mathbb{R})$. Thus we have

$$\begin{cases} -\mathrm{div}\left[\hat{U}_\varepsilon^2|\hat{v}_\varepsilon|^2\nabla(\phi_\varepsilon - \phi)\right] = \mathrm{div}\left[(\hat{U}_\varepsilon^2|\hat{v}_\varepsilon|^2 - a^2)\nabla\phi\right] & \text{in } B \\ \|\phi_\varepsilon - \phi\|_{L^2(B)} \to 0 \end{cases}.$$

From Theorem 2 in [22], there is $2 < q \le p$ and $C > 0$ s.t.

$$\|\nabla(\phi_\varepsilon - \phi)\|_{L^q(\tilde{B})} \le C\left(\tilde{R}^{-2+2/q}\|\phi_\varepsilon - \phi\|_{L^2(B)} + \|(\hat{U}_\varepsilon^2|\hat{v}_\varepsilon|^2 - a^2)\nabla\phi\|_{L^q(B)}\right) \underset{\varepsilon\to 0}{\to} 0.$$

Consequently, $\|\phi_\varepsilon - \phi\|_{W^{1,q}(\tilde{B})} \to 0$.

Step 3: We prove the third assertion

Let $\eta_0 > \eta > 0$, with η_0 defined by (46). Denote $d_k = \deg_{\partial B(\alpha_k,r)}(v_0)$. These integers do not depend on $r \in (\eta,\eta_0)$. Moreover, we have $\sum_k d_k = d_0$. For $r \in (\eta,\eta_0)$, we obtain that

$$2\pi|d_k| \le \int_{\partial B(\alpha_k,r)} |\partial_\tau v_0| \le \sqrt{2\pi r}\left(\int_{\partial B(\alpha_k,r)} |\partial_\tau v_0|^2\right)^{1/2},$$

and therefore

$$\frac{1}{2}\int_{B(\alpha_k,\eta_0)\setminus\overline{B(\alpha_k,\eta)}} |\nabla v_0|^2 \ge \pi d_k^2 \ln\frac{\eta_0}{\eta}.$$

Consequently, we have

$$\begin{aligned} \liminf \frac{1}{2}\int_{\cup_k B(\alpha_k,\eta_0)\setminus\overline{B(\alpha_k,\eta)}} |\nabla\hat{v}_\varepsilon|^2 &\ge \frac{1}{2}\int_{\cup_k B(\alpha_k,\eta_0)\setminus\overline{B(\alpha_k,\eta)}} |\nabla v_0|^2 \\ &\ge \pi\sum_k d_k^2 \ln\frac{\eta_0}{\eta}. \end{aligned} \tag{56}$$

By combining (47) and (56), we obtain the existence of C independent of ε and η s.t.

$$\begin{aligned} \frac{1}{2}\int_{\cup_k B(\alpha_k,\eta_0)} |\nabla\hat{v}_\varepsilon|^2 &\ge \pi\sum_k d_k^2\ln\frac{\eta_0}{\eta} + \pi d_0\ln\frac{\eta}{\xi} - C \\ &= \pi d_0\ln\frac{\eta_0}{\xi} + \pi(\sum_k d_k^2 - d_0)\ln\frac{\eta_0}{\eta} - C. \end{aligned}$$

Therefore, d_k must be either 0 or 1. Otherwise, (27) cannot hold for small η. Applying the strong convergence result from Step 2 with $K = B(\alpha_k,\eta) \setminus \overline{B(\alpha_k,\frac{\eta}{2})}$, we have that for small ε,

$$d_k = \deg_{\partial B(\alpha_k,\eta)}\left(\frac{\hat{v}_\varepsilon}{|\hat{v}_\varepsilon|}\right).$$

We next prove that $d_k = 1$ for each k. By contradiction, assume that there is k_0 s.t. $d_{k_0} = 0$. We may assume that $k_0 = 1$. From (43), there is a (separated) μ-bad disc $B(\hat{x}_0,\lambda\varepsilon^{1/4}/\delta)$ in $B(\alpha_1,\eta_0)$. Thus by (41), we have

$$\hat{F}_\xi(\hat{v}_\varepsilon, B(\hat{x}_0,\lambda\varepsilon^{1/4}/\delta)) > C(\mu)|\ln\varepsilon|.$$

On the other hand, since in $|\hat{v}_\varepsilon| \geq 1/2$ in $B(\alpha_k, \eta_0) \setminus \overline{B(\alpha_k, \eta_0/2)}$, applying Theorem 4.1 in [27] in $B(\alpha_k, \eta_0)$, $k \in \{2, ..., \kappa\}$, with $r = 10^{-4} \cdot \eta_0$ we find that

$$\hat{F}_\xi(\hat{v}_\varepsilon, B(\alpha_k, \eta_0)) \geq b^2 |\mathrm{deg}_{\partial B(\alpha_k,\eta_0)}(\hat{v}_\varepsilon)| |\ln \xi| - C, \ k = 2, ..., \kappa.$$

Since $\sum_{k=2}^{\kappa} \mathrm{deg}_{\partial B(\alpha_k,\eta_0)}(\hat{v}_\varepsilon) = d_0$, the above estimates yield

$$\hat{F}_\xi(\hat{v}_\varepsilon, B(0, \frac{\rho}{\delta})) \geq b^2 d_0 |\ln \xi| + C(\mu) |\ln \varepsilon| - C$$

which is in contradiction with (H) and (11). Thus $d_k = 1$ for $k \in \{1, ..., \kappa\}$ and consequently, $\kappa = d_0$. □

We are now in position to estimate the rate of uniform convergence of $|\hat{v}_\varepsilon|$ in a compact set $K \subset \mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_{d_0}\}$.

Corollary 4. *There is $C > 0$ s.t. for $\eta_0 > \eta > 0$ and small ε we have*

$$|\hat{v}_\varepsilon| \geq 1 - C |\ln \varepsilon|^{-1/3} \text{ in } B(0, \frac{\rho}{\delta}) \setminus \overline{B(\alpha_i, \eta)}.$$

Proof. Due to (36), it is sufficient to establish this result in $B(0, 1) \setminus \overline{B(\alpha_i, \eta)}$. Combining Corollary 3 with (44), we obtain that

$$\hat{F}_\xi(\hat{v}_\varepsilon, B(0, 2) \setminus \overline{B(\alpha_i, \eta/2)}) \leq C(\eta).$$

Thus for all $x \in B(0, \rho)$ s.t. $B(\hat{x}, \varepsilon^{1/4}/\delta) \subset B(0, 2) \setminus \overline{B(\alpha_i, \eta/2)}$, for small ε we have

$$F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4})) \leq \hat{F}_\xi(\hat{v}_\varepsilon, B(0, 2) \setminus \overline{B(\alpha_i, \eta/2)}) < |\ln \varepsilon|^{1/3}.$$

From Lemma 1 (first assertion), we obtain the existence of $C > 0$ (independent of ε and η) s.t. $|v_\varepsilon(x)| = |\hat{v}_\varepsilon(\hat{x})| \geq 1 - C |\ln \varepsilon|^{-1/3}$. Finally, since for all $\hat{x} \in B(0, 1) \setminus \overline{B(\alpha_i, \eta)}$ we have $B(\hat{x}, \varepsilon^{1/4}/\delta) \subset B(0, 2) \setminus \overline{B(\alpha_i, \eta/2)}$, Corollary 4 follows. □

3.6 Information about the limit v_0

Following [11] (Appendix IV, page 152) we have

Proposition 9. *For all $1 \leq p < 2$ and for any compact $K \subset \mathbb{R}^2$, $\hat{v}_\varepsilon$ is bounded in $W^{1,p}(K)$.*

Let θ_i be the main argument of $\frac{\hat{x} - \alpha_i}{|\hat{x} - \alpha_i|}$ and set $\theta = \theta_1 + ... + \theta_{d_0}$. Note that $\nabla \theta$ is smooth away from $\{\alpha_1, ..., \alpha_\kappa\}$ and $\Pi_i \frac{\hat{x} - \alpha_i}{|\hat{x} - \alpha_i|} = \mathrm{e}^{\imath \theta}$. Let $\tilde{g} := \mathrm{tr}_{\partial B_1} v_0$ and $\varphi_0 \in C^\infty(\partial B_1, \mathbb{R})$ be s.t. $\tilde{g} = \Pi_i \frac{\hat{x} - \alpha_i}{|\hat{x} - \alpha_i|} \mathrm{e}^{\imath \varphi_0} = \mathrm{e}^{\imath(\theta + \varphi_0)}$ (see [12] for the existence of φ_0).

Proposition 10. *The limit v_0 satisfies $-\mathrm{div}\left(a^2 \, v_0 \times \nabla v_0\right) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Moreover we may write $v_0 = \mathrm{e}^{\imath(\theta + \varphi_\star)}$. Here $\varphi_\star$ is the solution of*

$$\begin{cases} -\mathrm{div}\left[a^2 \nabla(\theta + \varphi_\star)\right] = 0 & \text{in } B_1 \\ \varphi_\star = \varphi_0 & \text{on } \partial B_1 \end{cases}. \tag{57}$$

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^2)$, and set $K = \mathrm{supp}(\phi)$. By Proposition 9, we have $\hat{U}_\varepsilon^2 \, \hat{v}_\varepsilon \times \nabla \hat{v}_\varepsilon \rightharpoonup a^2 \, v_0 \times \nabla v_0$ in $L^p(K)$ for $p < 2$. Multiplying the equation $-\mathrm{div}\left[\hat{U}_\varepsilon^2 \, \hat{v}_\varepsilon \times \nabla \hat{v}_\varepsilon\right] = 0$ by ϕ and integrating by parts, we obtain

$$\begin{aligned} 0 = \int_K -\mathrm{div}\left[\hat{U}_\varepsilon^2 \, \hat{v}_\varepsilon \times \nabla \hat{v}_\varepsilon\right] \phi &= \int_K \hat{U}_\varepsilon^2 \, \hat{v}_\varepsilon \times \nabla \hat{v}_\varepsilon \cdot \nabla \phi \\ &\to \int_K a^2 \, v_0 \times \nabla v_0 \cdot \nabla \phi = \int_K -\mathrm{div}\left(a^2 \, v_0 \times \nabla v_0\right) \phi. \end{aligned}$$

Consequently $-\text{div}\left(a^2\, v_0 \times \nabla v_0\right) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$.

In order to prove that $-\text{div}\left[a^2 \nabla(\theta + \varphi_\star)\right] = 0$ in $B_1 \setminus \{\alpha_1, ..., \alpha_{d_0}\}$, we follow [18], Step 12 of Theorem C.

Next, we prove that $\varphi_\star$ is harmonic in a neighborhood of α_k. Fix $\lambda > 0$ and $x_0 \in \omega$ s.t. $B(x_0, 2\lambda) \subset \omega \setminus \{\alpha_1, ..., \alpha_{d_0}\}$. As we established in Proposition 8, Step 2, $F_\xi(\hat{v}_\varepsilon, B(x_0, 2\lambda))$ is uniformly bounded in ε. Proceeding as in Step 2, we conclude that exists $\lambda_0 \in (\lambda, 2\lambda)$ s.t., after passing to a further subsequence, we have

$$\int_{\partial B(x_0,\lambda_0)} \left\{ |\nabla \hat{v}_\varepsilon|^2 + \frac{1}{2\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2 \right\} \leq C \tag{58}$$

with C and λ_0 independent of ε. Now, if $\hat{u}_\varepsilon$ minimizes

$$\hat{E}_\xi(\hat{u}) = \frac{1}{2} \int_{B(0,\frac{\varrho}{\delta})} \left\{ |\nabla \hat{u}|^2 + \frac{1}{2\xi^2}(a^2 - |\hat{u}|^2)^2 \right\}$$

subject to $\hat{u}(x) = f_\varepsilon(\delta x)$ on $\partial B(0, \frac{\varrho}{\delta})$, then $\hat{u}_\varepsilon$ minimizes $\hat{E}_\xi(\hat{u}, B(x_0, \lambda_0))$ with respect to its own boundary conditions. In other words, $\hat{w}_\varepsilon := \frac{\hat{u}_\varepsilon}{b}$ minimizes the classical energy

$$\frac{1}{2} \int_{B(x_0,\lambda_0)} \left\{ |\nabla \hat{w}_\varepsilon|^2 + \frac{b^2}{2\xi^2}(1 - |\hat{w}_\varepsilon|^2)^2 \right\}$$

among $w \in H^1(B(x_0, \lambda_0))$ such that $w = h_\varepsilon := \frac{\hat{u}_\varepsilon}{b}$ on $\partial B(x_0, \lambda_0)$. It follows from (58) and Proposition 2 that h_ε also satisfies

$$\int_{\partial B(x_0,\lambda_0)} \left\{ |\partial_\tau h_\varepsilon|^2 + \frac{1}{2\xi^2}(1 - |h_\varepsilon|^2)^2 \right\} \leq C + 1. \tag{59}$$

Note that by Proposition 2 we have

$$\|\hat{w}_\varepsilon\|_{L^\infty(B(x_0,\lambda_0))} \leq 1 + ce^{-c_0 \xi}. \tag{60}$$

Using (60) and the uniform bound from Corollary 4, we may repeat the arguments of Theorem 2 in [10] and conclude that, up to a subsequence, there exists an $\mathbb{S}^1$-valued map w_0 s.t. for every compact $K \subset (\omega \setminus \{\alpha_1, ...\alpha_{d_0}\})$ we have

$$\hat{w}_\varepsilon \to w_0 \text{ in } C^\infty(K) \tag{61}$$

and

$$\frac{b^2(1 - |\hat{w}_\varepsilon|^2)}{\xi^2} \to |\nabla w_0|^2 \text{ in } C^\infty(K). \tag{62}$$

Fix now $r < \min\left\{ \dfrac{\min|\alpha_k - \alpha_j|}{8}, \dfrac{\text{dist}(\alpha_k, \partial\omega)}{8} \right\}$ and denote $\omega_r := \{x \in \omega, \text{dist}(x, \partial\omega) > r\}$. It follows from (61) that $\hat{w}_\varepsilon \to q_0 := \text{tr}_{\partial\omega_r} w_0$ in $C^\infty(\partial\omega_r)$. In view of Proposition 9, we have $w_0 \in W^{1,p}(\omega_r)$, $p < 2$. By Remark I.1 in [11], this implies that

$$w_0 = \tilde{w} \exp\left(i \sum_k c_k \ln|x - \alpha_k| + i\chi \right).$$

Here:

- $\tilde{w}$ is the *canonical harmonic map* (see [11], Sec. I.3.) having singularities $\{\alpha_k, k = 1, ..., d_0\}$ and equal to q_0 on $\partial\omega_r$;

- the c_k's are real coefficients;
- χ is the solution of

$$\begin{cases} \Delta\chi = 0 & \text{in } \omega_r \\ \chi(x) + \sum_k c_k \ln|x-\alpha_k| = 0 & \text{on } \partial\omega_r \end{cases}.$$

Repeating the argument of [11], Theorem VII.1, Step 2 (the key ingredients of this proof are (61), (62) and Corollary 3), we find that $c_k \equiv 0, k = 1, ..., d_0$, and, consequently, $w_0 \equiv \tilde{w}$ in ω_r. Finally, by [11], Corollary I.2., we know that the canonical harmonic map $\tilde{w}$ is of the form $\tilde{w} = e^{i(\theta+\varphi_\star)}$ with $\varphi_\star$ harmonic in ω_r.

□

3.7 Uniqueness of zeros

Proposition 11. *For ε sufficiently small, the minimizer $\hat{v}_\varepsilon$ has exactly d_0 zeros.*

Proof. It suffices to prove that for small ε there is a unique zero of $\hat{v}_\varepsilon$ in $B(\alpha_k, r)$, $k = 1, ..., d_0$, with r defined in the proof of Proposition 10.

Since $\hat{w}_\varepsilon = \frac{\hat{v}_\varepsilon \hat{U}_\varepsilon}{b}$, from Proposition 2 and Proposition 10 we see that $w_0 = v_0 = e^{i(\theta_k+H_k)}$ in $B(\alpha_k, r)$, where θ_k is the phase of $\frac{x-\alpha_k}{|x-\alpha_k|}$ and $H_k = \varphi_\star + \psi_k$ is harmonic in $B(\alpha_k, r)$. Using (61) and (62) and arguing as in the alternative proof of Theorem VII.4 in [11] (page 74) we obtain that $\nabla H_k(\alpha_k) = 0$.

Finally, we are now in position to obtain, as in Theorem IX.1 [11] (using the main result of [7]), that there is a unique zero of $\hat{w}_\varepsilon$ (and, therefore, of $\hat{v}_\varepsilon$) in $B(\alpha_k, r)$.

□

3.8 Summary

We have thus proved

Theorem 3. *Let $\varepsilon_n \downarrow 0$ and $\hat{v}_{\varepsilon_n}$ be a minimizer of (17) in (18) for $\varepsilon = \varepsilon_n$. Then there exist d_0 distinct points $\alpha_1, ..., \alpha_{d_0} \in \omega$ and a function $v_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_{d_0}\}, \mathbb{S}^1) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{S}^1)$ ($p < 2$) s.t., up to a subsequence*

1. *$\hat{v}_{\varepsilon_n} \to v_0$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_{d_0}\})$ and $C^0_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_{d_0}\})$,*
2. *$\hat{v}_{\varepsilon_n} \rightharpoonup v_0$ in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ ($p < 2$),*
3. *for $K \Subset \mathbb{R}^2 \setminus \{\alpha_1, ..., \alpha_{d_0}\}$, $|\hat{v}_{\varepsilon_n}| \geq 1 - |\ln\varepsilon_n|^{-1/3}$ in K and $\displaystyle\int_K |\nabla|\hat{v}_{\varepsilon_n}||^2 + \frac{1}{\xi^2}(1-|\hat{v}_{\varepsilon_n}|^2)^2 \to 0$,*
4. *for $K \Subset \mathbb{R}^2 \setminus \overline{\omega}$, $\hat{v}_{\varepsilon_n} \to v_0$ in $C^\infty(K)$ and $1 - |\hat{v}_{\varepsilon_n}| \leq C_K \xi^2$,*
5. *$\hat{v}_{\varepsilon_n}$ has exactly d_0 zeros $x^n_1, ..., x^n_{d_0}$ and $x^n_i \to \alpha_i$,*
6. *v_0 satisfies $-\mathrm{div}\left(a^2\, v_0 \times \nabla v_0\right) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$.*

Let us summarize the proof of Theorem 3:

- Statement *1.* is established in Proposition 8,
- Statement *2.* follows from Propositions 9 and 10,
- Statement *3.* is a consequence of Corollary 4 and (54),
- Statement *4.* is Corollary 2,

- Statement *5.* is proved in Proposition 11,
- Statement *6.* is established in Proposition 10.

The proof of Theorem 3 is complete.

4 Renormalized energy for the model problem

In this section, we establish the expansion for $F_\varepsilon(v_\varepsilon, B(0,\rho)) = \hat{F}_\xi(\hat{v}_\varepsilon, B(0, \frac{\rho}{\delta}))$; specifically, we derive the expression for

$$\lim_{\varepsilon} \left\{ \hat{F}_\xi(\hat{v}_\varepsilon, B(0, \frac{\rho}{\delta})) - \pi d_0^2 |\ln \delta| - \pi d_0 b^2 |\ln \xi| \right\}. \tag{63}$$

(We are going to prove that this limit exists).

In order to find an expression for (63), our strategy is the following:

- (Section 4.1) We first study the minimization of the Dirichlet functional among $\mathbb{S}^1$-valued maps in annulars $B(0,\rho/\delta) \setminus \overline{B(0,1)}$ with the Dirichlet boundary conditions: $f^\delta(\delta \cdot)$ on $\partial B(0,\rho/\delta)$ and g^δ on $\partial B(0,1)$. Here $f^\delta = \frac{f_\varepsilon}{|f_\varepsilon|}$ where f_ε is given in the model problem and $g^\delta, g^0 \in C^\infty(\partial B_1, \mathbb{S}^1)$ are s.t. $g^\delta \to g^0$ in C^1. We get that the Dirichlet energy has the form $\pi d_0^2 \ln(\rho/\delta) + \tilde{W}_0(f_0) + \tilde{W}_1(g^0) + o_\delta(1)$.
- (Sections 4.2, 4.3 and 4.4) In $B(0,1)$, we study the weighted Ginzburg-Landau functional with the Dirichlet boundary condition g^δ on $\partial B(0,1)$. Making use of the previous bullet point, one may obtain the matching upper and lower bounds and use them to derive the third term of renormalized energy, which depends on the limiting locations of the zeros $\boldsymbol{\beta} = (\beta_1, ..., \beta_{d_0}) \in \omega^{d_0}$ and on g^0. We establish that

$$\inf_{H^1_{g^\delta}} \hat{F}_\xi(\cdot, B(0,1)) = \pi d_0 b^2 \ln \frac{b}{\xi} + d_0 b^2 \gamma + \tilde{W}_2(\boldsymbol{\beta}, g^0) + o_\varepsilon(1).$$

- (Section 4.5) Finally, we make a fundamental observation: the limiting function $g_0 = \lim \mathrm{tr}_{\partial B_1} \hat{v}_\varepsilon$ and the points $\boldsymbol{\alpha}$ obtained form Theorem 3 form a minimal configuration for $W_1(\tilde{g}) + W_2(\boldsymbol{\beta}, \tilde{g})$. Thus, introducing

$$\tilde{W}(\boldsymbol{\beta}) = \inf_{\substack{\tilde{g} \in C^\infty(\partial B_1, \mathbb{S}^1) \\ \text{with } \deg_{\partial B_1}(\tilde{g}) = d_0}} \left\{ \tilde{W}_1(\tilde{g}) + \tilde{W}_2(\boldsymbol{\beta}, \tilde{g}) \right\}$$

  we conclude that $\boldsymbol{\alpha}$ minimizes $\tilde{W}$.

In this section we prove the following theorem.

Theorem 4. *The following energy expansion holds when $\varepsilon \to 0$*

$$\hat{F}_\xi(\hat{v}_\varepsilon, B(0, \frac{\rho}{\delta})) = \pi d_0 b^2 \ln \frac{b}{\xi} + \pi d_0^2 \ln \frac{\rho}{\delta} + \tilde{W}_0(f_0) + \tilde{W}(\boldsymbol{\alpha}) + d_0 b^2 \gamma + o_\varepsilon(1). \tag{64}$$

Here the points $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_{d_0})$ are obtained from Theorem 3, $\gamma > 0$ is an absolute constant and $\tilde{W}_0(f_0), \tilde{W}(\boldsymbol{\alpha})$ are renormalized energies:

- *$\tilde{W}_0$ is independent of the points $\alpha_1, ..., \alpha_{d_0}$ and given by (72),*
- *$\tilde{W}$ is given by* (90), *it is independent of f_0 and the limiting points $(\alpha_1, .., \alpha_{d_0})$ minimize $\tilde{W}$.*

Remark 2. The renormalized energy in the expansion (64) decouples into the part that depends only on the external boundary conditions $\tilde{W}_0(f_0)$ and the part that depends only on the location of the vortices $\tilde{W}(\boldsymbol{\alpha})$. Since $\boldsymbol{\alpha}$ minimizes $\tilde{W}$, the external boundary data has no effect on the location of vortices inside the inclusion. This is a drastic difference with the results of [11] and [18], where the Dirichlet boundary data on the external boundary influences the location of the vortices.

4.1 Minimization among $\mathbb{S}^1$-valued maps away from the inclusion

Denote $B_\rho := B(0,\rho)$. Let

$$(f^\delta)_{0<\delta<1} \subset C^\infty(\partial B_\rho,\mathbb{S}^1),\ f^0 \in C^\infty(\partial B_\rho,\mathbb{S}^1) \text{ be s.t. } \begin{cases} f^\delta \underset{\delta\to 0}{\to} f^0 \text{ in } C^1(\partial B_\rho) \\ \deg_{\partial B_\rho}(f^\delta) = d_0 \end{cases},$$

and

$$(g^\delta)_{0<\delta<1} \subset C^\infty(\partial B_1,\mathbb{S}^1),\ g^0 \in C^\infty(\partial B_1,\mathbb{S}^1) \text{ be s.t. } \begin{cases} g^\delta \underset{\delta\to 0}{\to} g^0 \text{ in } C^1(\partial B_1) \\ \deg_{\partial B_1}(g^\delta) = d_0 \end{cases}.$$

For $\delta \in (0,1)$, we denote $A_\delta = B_{\rho/\delta} \setminus \overline{B_1}$ and

$$W_\delta = \{u \in H^1(A_\delta,\mathbb{S}^1)\,|\,\mathrm{tr}_{\partial B_{\rho/\delta}} u(\cdot) = f^\delta(\delta\cdot) \text{ and } \mathrm{tr}_{\partial B_1} u = g^\delta\},$$

$$Y_\delta = \{u \in H^1(A_\delta,\mathbb{S}^1)\,|\,\mathrm{tr}_{\partial B_{\rho/\delta}} u(\cdot) = f^0(\delta\cdot) \text{ and } \mathrm{tr}_{\partial B_1} u = g^0\}.$$

Consider the following minimization problems:

$$I_\delta(f^\delta,g^\delta) = I_\delta = \inf_{u\in W_\delta} \frac{1}{2}\int_{A_\delta} |\nabla u|^2. \tag{P_δ}$$

$$J_\delta(f^0,g^0) = J_\delta = \inf_{u\in Y_\delta} \frac{1}{2}\int_{A_\delta} |\nabla u|^2. \tag{Q_δ}$$

Proposition 12. *For small ε, I_δ is close to J_δ, namely*

$$I_\delta = J_\delta + o_\delta(1). \tag{65}$$

Proof. In this subsection θ stands for the main argument of z *i.e.* $\frac{z}{|z|} = \mathrm{e}^{\imath\theta}$. For $\delta \geq 0$, let $\phi_\delta \in C^\infty(\partial B_1,\mathbb{R})$ be s.t. $g^\delta = \mathrm{e}^{\imath(d_0\theta+\phi_\delta)}$ and $\zeta_\delta \in C^\infty(\partial B_\rho,\mathbb{R})$ be s.t. $f^\delta = \mathrm{e}^{\imath(d_0\theta+\zeta_\delta)}$. We may assume that $\phi_\delta \to \phi_0$ in $C^1(\partial B_1)$ and $\zeta_\delta \to \zeta_0$ in $C^1(\partial B_\rho)$. Note that

$$u \in W_\delta \iff u = \mathrm{e}^{\imath(\varphi+d_0\theta)} \text{ with } \varphi \in w_\delta. \tag{66}$$

Here $w_\delta := \{\varphi \in H^1(A_\delta,\mathbb{R})\,|\,\mathrm{tr}_{\partial B_{\frac{\rho}{\delta}}}\varphi(\cdot) = \zeta_\delta(\delta\cdot) \text{ and } \mathrm{tr}_{\partial B_1}\varphi = \phi_\delta\}$.

Since $\Delta\theta = 0$ in A_δ and $\partial_\nu\theta = 0$ on ∂A_δ, for $u \in W_\delta$ we have

$$\int_{A_\delta} |\nabla u|^2 = \int_{A_\delta} |\nabla(\varphi + d_0\theta)|^2 = d_0^2\int_{A_\delta} |\nabla\theta|^2 + \int_{A_\delta} |\nabla\varphi|^2.$$

Consequently, the problem (P_δ) has a unique solution $u_\delta = \mathrm{e}^{\imath(d_0\theta+\varphi_\delta)}$, with φ_δ being the unique solution of

$$\begin{cases} -\Delta\varphi_\delta = 0 & \text{in } A_\delta \\ \varphi_\delta(\cdot) = \zeta_\delta\,(\delta\cdot) & \text{on } \partial B_{\frac{\rho}{\delta}} \\ \varphi_\delta = \phi_\delta & \text{on } \partial B_1 \end{cases}.$$

With the same argument, the problem (Q_δ) admits a unique solution $v_\delta = \mathrm{e}^{\imath(d_0\theta+\psi_\delta)}$ with ψ_δ being the unique solution of

$$\begin{cases} -\Delta\psi_\delta = 0 & \text{in } A_\delta \\ \psi_\delta(\cdot) = \zeta_0\,(\delta\cdot) & \text{on } \partial B_{\frac{\rho}{\delta}} \\ \psi_\delta = \phi_0 & \text{on } \partial B_1 \end{cases}.$$

Denote $\eta_\delta = \varphi_\delta - \psi_\delta$. Then η_δ is the unique solution of

$$\begin{cases} \Delta\eta_\delta = 0 & \text{in } A_\delta \\ \eta_\delta = \hat{\zeta}_\delta - \hat{\zeta}_0 & \text{on } \partial B_{\frac{\rho}{\delta}} \\ \eta_\delta = \phi_\delta - \phi_0 & \text{on } \partial B_1 \end{cases}.$$

(Here $\hat{\zeta}(x) := \zeta(\delta x)$).

One may prove that $\|\psi_\delta\|_{L^2(A_\delta)}$ is bounded and more precisely we have the following result.

Proposition 13.

$$\frac{1}{2}\int_{A_\delta} |\nabla\psi_\delta|^2 \to |\phi_0|^2_{H^{1/2}(\mathbb{S}^1)} + |\zeta_0|^2_{H^{1/2}(\partial B_\rho)}, \ \ \textit{as } \delta \to 0. \tag{67}$$

Proof. Let $(a_n)_{n\in\mathbb{Z}}, (b_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ be s.t.

$$\phi_0(\mathrm{e}^{\imath\theta}) = \sum_{n\in\mathbb{Z}} a_n \mathrm{e}^{\imath n\theta} \text{ and } \zeta_0(\rho\mathrm{e}^{\imath\theta}) = \sum_{n\in\mathbb{Z}} b_n \mathrm{e}^{\imath n\theta}.$$

We have

$$|\phi_0|^2_{H^{1/2}(\mathbb{S}^1)} = \sum_{\mathbb{Z}} |n||a_n|^2 \text{ and } |\zeta_0|^2_{H^{1/2}(\partial B_\rho)} = |\hat{\zeta}_0|^2_{H^{1/2}(\partial B_{\frac{\rho}{\delta}})} = \sum_{\mathbb{Z}} |n||b_n|^2.$$

From [8] (Appendix D.), denoting $R(\delta) = \frac{\rho}{\delta}$, we have

$$\begin{aligned} \frac{1}{2\pi}\int_{A_\delta} |\nabla\psi_\delta|^2 &= \frac{|b_0 - a_0|^2}{\ln R(\delta)} + \sum_{n\neq 0} \frac{|n|}{R(\delta)^{2|n|} - 1}\Big[(|a_n|^2 + |b_n|^2)(R(\delta)^{2|n|} + 1) \\ &\qquad - 2(\overline{a_n}b_n + a_n\overline{b_n})R(\delta)^{|n|}\Big] \\ &= |\phi_0|^2_{H^{1/2}(\partial B_1)} + |\psi|^2_{H^{1/2}(\partial B_\rho)} + \frac{|b_0 - a_0|^2}{\ln R(\delta)} \\ &\qquad + \sum_{n\neq 0} \frac{2}{R(\delta)^{2|n|} - 1}\Big[(|a_n|^2 + |b_n|^2) - (\overline{a_n}b_n + a_n\overline{b_n})R(\delta)^{|n|}\Big] \\ &= |\phi_0|^2_{H^{1/2}(\partial B_1)} + |\zeta_0|^2_{H^{1/2}(\partial B_\rho)} + o_\delta(1). \end{aligned}$$

Consequently, as $\delta \to 0$, we obtain (67). ☐

Following the same lines as Proposition 13 we obtain

$$\|\nabla\varphi_\delta\|_{L^2(A_\delta)} \le C \text{ with } C \text{ independent of } \delta, \tag{68}$$

and

$$\|\nabla\eta_\delta\|_{L^2(A_\delta)} \to 0. \tag{69}$$

It follows from (68) and (69) that

$$\begin{aligned} I_\delta &= \frac{d_0^2}{2}\int_{A_\delta} |\nabla\theta|^2 + \frac{1}{2}\int_{A_\delta} |\nabla\varphi_\delta|^2 \\ &= \frac{d_0^2}{2}\int_{A_\delta} |\nabla\theta|^2 + \frac{1}{2}\int_{A_\delta} |\nabla\psi_\delta|^2 + \int_{A_\delta} \nabla\psi_\delta\cdot\nabla\eta_\delta + \frac{1}{2}\int_{A_\delta} |\nabla\eta_\delta|^2 \\ &= J_\delta + o_\delta(1). \end{aligned} \tag{70}$$

☐

From (70) and (67), we deduce that

$$I_\delta = J_\delta + o_\delta(1) = \pi d_0^2 \ln\frac{\rho}{\delta} + \tilde{W}_0(f^0) + \tilde{W}_1(g^0) + o_\delta(1) \tag{71}$$

with

$$\tilde{W}_0(f_0) = |\zeta_0|^2_{H^{1/2}(\partial B_\rho)} \text{ and } \tilde{W}_1(g^0) = |\phi_0|^2_{H^{1/2}(\partial B_1)}. \tag{72}$$

One of the main ingredients in the study of the renormalized energy is that the Dirichlet condition $f_{\min}(x) = \gamma_0 \frac{x^{d_0}}{|x|^{d_0}}$, $\gamma_0 \in \mathbb{S}^1$ minimizes W_0. More precisely, for all $f_0 \in C^1(\partial B_1, \mathbb{S}^1)$ s.t. $\deg_{\partial B_1}(f_0) = d_0$, we have

$$W_0(f_{\min}) = 0 \leq W_0(f_0). \tag{73}$$

4.2 Energy estimates for $\mathbb{S}^1$-valued maps around the inclusion

Let $g^0 \in C^\infty(\partial B_1, \mathbb{S}^1)$ be s.t. $\deg_{\partial B_1}(g^0) = d_0 > 0$, $\beta_1, ..., \beta_{d_0}$ are d_0 distinct points of ω,

$$\eta_0 := \frac{1}{4}\min_i \left\{ \mathrm{dist}(\beta_i, \partial\omega), \min_{j\neq i} |\beta_i - \beta_j| \right\}.$$

For $r \in (0, \eta_0)$, we define

$$\Omega_r := B_1 \setminus \cup_k \overline{B(\beta_k, r)},$$

$$\mathcal{E}_r := \{u \in H^1(\Omega_r, \mathbb{S}^1) \,|\, \mathrm{tr}_{\partial B_1} u = g^0 \text{ and } \deg_{\partial B(\beta_i, r)}(u) = 1\}$$

and

$$\mathcal{F}_r := \left\{ u \in H^1(\Omega_r, \mathbb{S}^1) \,|\, \mathrm{tr}_{\partial B_1} u = g^0 \text{ and there are } \gamma_i \in \mathbb{S}^1 \text{ s.t. } \mathrm{tr}_{\partial B(\beta_i, r)} u(x) = \gamma_i \frac{x - \beta_i}{|x - \beta_i|} \right\}.$$

Consider two minimization problems

$$K(r, g^0, \boldsymbol{\beta}) = K(r) = \inf_{u \in \mathcal{E}_r} \frac{1}{2}\int_{\Omega_r} a^2 |\nabla u|^2 \tag{R_r}$$

and

$$L(r, g^0, \boldsymbol{\beta}) = L(r) = \inf_{u \in \mathcal{F}_r} \frac{1}{2}\int_{\Omega_r} a^2 |\nabla u|^2, \, \boldsymbol{\beta} = \{\beta_1, ..., \beta_{d_0}\}. \tag{S_r}$$

We denote $\theta = \theta_1 + ... + \theta_{d_0}$ where θ_i is the main argument of $\frac{x - \beta_i}{|x - \beta_i|}$, *i.e.*, $\frac{x - \beta_i}{|x - \beta_i|} = \mathrm{e}^{\imath\theta_i}$.

Let ψ_0 be the unique (up to an additive constant in $2\pi\mathbb{Z}$) solution of

$$\begin{cases} -\mathrm{div}\left[a^2(\nabla\psi_0 + \nabla\theta)\right] = 0 & \text{in } B_1 \\ \mathrm{e}^{\imath(\theta + \psi_0)} = g^0 & \text{on } \partial B_1 \end{cases}. \tag{74}$$

Lemma 4. ([18], Appendix A.)

$$K(r) = \frac{1}{2}\int_{\Omega_r} a^2 |\nabla\theta + \nabla\psi_0|^2 + \mathcal{O}(r|\ln r|),$$

$$L(r) = \frac{1}{2}\int_{\Omega_r} a^2 |\nabla\theta + \nabla\psi_0|^2 + \mathcal{O}(r|\ln r|),$$

with

$$\frac{1}{2}\int_{\Omega_r} a^2 |\nabla\theta + \nabla\psi_0|^2 = \pi d_0 b^2 |\ln r| + \tilde{W}_2(\boldsymbol{\beta}, g^0) + \mathcal{O}(r^2). \tag{75}$$

In (75), $\tilde{W}_2(\boldsymbol{\beta}, g^0)$, whose explicit expression is given in [18], formula (106), depends only on $\boldsymbol{\beta}$ and g^0.

4.3 Upper bound for the energy

Lemma 5. *Fix $\rho > 0$ and let $f_\varepsilon \in C^\infty(\partial B_\rho)$, $f_0 \in C^\infty(\partial B_\rho, \mathbb{S}^1)$ be s.t. $f_\varepsilon \to f_0$ in $C^1(\partial B_\rho)$. Let $\boldsymbol{\beta} = (\beta_1, ..., \beta_{d_0}) \in \omega^{d_0}$ be s.t. $\beta_i \neq \beta_j$ for $i \neq j$. Then, for each $g^0 \in C^\infty(\partial B_1, \mathbb{S}^1)$, the following upper bound holds:*

$$\inf_{H^1_{f_\varepsilon}(B_\rho)} F_\varepsilon \leq \pi d_0 b^2 \ln \frac{b}{\xi} + \pi d_0^2 \ln \frac{\rho}{\delta} + \tilde{W}_0(f_0) + \tilde{W}_1(g^0) + \tilde{W}_2(\boldsymbol{\beta}, g^0) + d_0 b^2 \gamma + o_\varepsilon(1). \tag{76}$$

Here $\tilde{W}_0$, $\tilde{W}_1$ are defined by (72) *and $\tilde{W}_2$ by* (75)*.*

Proof. We construct a test function $w_\varepsilon \in H^1_{\hat{f}_\varepsilon}(B_{\rho/\delta}, \mathbb{C})$ which gives (76). Fix $0 < r < \eta_0$. Let

$$u_\delta \text{ be the minimizer of } (P_\delta) \text{ with } g^\delta \equiv g^0 \text{ and } f^\delta = \frac{f_\varepsilon}{|f_\varepsilon|}$$

and

$$u_r \text{ be the minimizer of } (S_r).$$

Note that $f^\delta \to f_0 = \lim_\varepsilon f_\varepsilon$ in $C^1(\partial B_\rho)$. For each $i = 1, ..., d_0$ let $u_i^{\xi,r}$ be the global minimizer of the classic Ginzburg-Landau energy in $B(\beta_i, r)$ with the parameter ξ/b and the boundary condition $u_i^{\xi,r}(x) = h_i^r(x) := \gamma_i \frac{x - \beta_i}{r}$ on $\partial B(\beta_i, r)$, $\gamma_i \in \mathbb{S}^1$ is defined through u_r. Denote

$$\begin{aligned} I(\xi/b, r) &:= \inf_{H^1_{h_i^r}(B(\beta_i, r))} \frac{1}{2} \int_{B(\beta_i, r)} \left\{ |\nabla u|^2 + \frac{b^2}{2\xi^2}(1 - |u|^2)^2 \right\} \\ &= \frac{1}{2} \int_{B(\beta_i, r)} \left\{ |\nabla u_i^{\xi,r}|^2 + \frac{b^2}{2\xi^2}(1 - |u_i^{\xi,r}|^2)^2 \right\}. \end{aligned} \tag{77}$$

Lemma IX.1 in [11] implies that

$$I(\xi/b, r) = \pi \ln \frac{br}{\xi} + \gamma + o_\xi(1). \tag{78}$$

We next extend the u_i's to $B_{\rho/\delta}$. For this purpose, we consider $\zeta \in C^\infty(\mathbb{R}, [0, 1])$ s.t. $\zeta = 0$ in $\mathbb{R}^-$ and $\zeta = 1$ in $[1, \infty)$ and set

$$\chi_\varepsilon(s e^{\imath\theta}) = \zeta\left(s - \frac{\rho}{\delta} + 1\right)\Big[|f_\varepsilon|(\rho e^{\imath\theta}) - 1\Big] + 1.$$

In view of (14), we have $\|\chi_\varepsilon - 1\|_{L^2(B_{\rho/\delta})} \leq C\varepsilon$. Consider the following test function

$$\tilde{u} = \begin{cases} \chi_\varepsilon u_\delta & \text{in } B_{\rho/\delta} \setminus \overline{B_1} \\ u_r & \text{in } B_1 \setminus \cup \overline{B(\beta_i, r)} \\ u_i^{\xi,r} & \text{in } B(\beta_i, r) \end{cases}. \tag{79}$$

Clearly,

$$\inf_{H^1_{f_\varepsilon}} F_\varepsilon \leq \hat{F}_\xi(w_\varepsilon) \leq \pi d_0^2 \ln \frac{\rho}{\delta} + \tilde{W}_1(g^0) + \tilde{W}_2(\boldsymbol{\beta}, g^0) + \tilde{W}_0(f_0) + \pi d_0 b^2 \ln \frac{b}{\xi} + d_0 b^2 \gamma + o_\varepsilon(1) + h(r)$$

with $h(r) = o_r(1)$. Thus, letting $r \to 0$ as $\varepsilon \to 0$ we obtain the desired upper bound. □

4.4 Lower bound

We prove that the upper bound (76) is sharp by constructing the matching lower bound.

Lemma 6. *Let $\varepsilon_n \downarrow 0$, $\hat{v}_{\varepsilon_n}$ be a minimizer of (17) in (18) for $\varepsilon = \varepsilon_n$ and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_{d_0}) \in \omega^{d_0}$ be given by Theorem 3. Denote $g_0 := \lim \mathrm{tr}_{\partial B_1} \hat{v}_\varepsilon \in C^\infty(\partial B_1, \mathbb{S}^1)$. Then the following lower bound holds:*

$$F(\hat{v}_\varepsilon, B_{\frac{\rho}{\delta}}) \geq \pi d_0 b^2 \ln \frac{b}{\xi} + \pi d_0^2 \ln \frac{\rho}{\delta} + \tilde{W}_0(f_0) + \tilde{W}_1(g_0) + \tilde{W}_2(\boldsymbol{\alpha}, g_0) + d_0 b^2 \gamma + o_\varepsilon(1). \tag{80}$$

Proof. As in the proof of Lemma 5, we split $B_{\frac{\rho}{\delta}}$ into three parts: $B_{\frac{\rho}{\delta}} \setminus \overline{B_1}$, $B_1 \setminus \cup \overline{B(\alpha_i, r)}$ and $\cup B(\alpha_i, r)$ with small $0 < r < \eta_0$.

In $B_{\frac{\rho}{\delta}} \setminus \cup \overline{B(\alpha_i, r)}$ one may write $\hat{v}_\varepsilon = |\hat{v}_\varepsilon| w_\varepsilon$. Using Corollary 4 and (44) we have

$$\begin{aligned} \hat{F}_\xi(\hat{v}_\varepsilon, B_{\frac{\rho}{\delta}} \setminus \overline{B_1}) &= \frac{1}{2} \int_{B_{\frac{\rho}{\delta}} \setminus \overline{B_1}} \left\{ \hat{U}_\varepsilon^2 |\hat{v}_\varepsilon|^2 |\nabla w_\varepsilon|^2 + \hat{U}_\varepsilon^2 |\nabla |\hat{v}_\varepsilon||^2 + \frac{\hat{U}_\varepsilon^4}{2\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2 \right\} \\ &= \frac{1}{2} \int_{B_{\frac{\rho}{\delta}} \setminus \overline{B_1}} \left\{ \hat{U}_\varepsilon^2 |\nabla w_\varepsilon|^2 + \hat{U}_\varepsilon^2 |\nabla |\hat{v}_\varepsilon||^2 + \frac{\hat{U}_\varepsilon^4}{2\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2 \right\} + o_\varepsilon(1) \\ &\geq \frac{1}{2} \int_{B_{\frac{\rho}{\delta}} \setminus \overline{B_1}} \hat{U}_\varepsilon^2 |\nabla w_\varepsilon|^2 + o_\varepsilon(1). \end{aligned} \tag{81}$$

We take $g^\delta = \dfrac{\mathrm{tr}_{\partial B_1} \hat{v}_\varepsilon}{|\mathrm{tr}_{\partial B_1} \hat{v}_\varepsilon|}$ and $f^\delta = \dfrac{\mathrm{tr}_{\partial B_\rho} v_\varepsilon}{|\mathrm{tr}_{\partial B_\rho} v_\varepsilon|}$. Note that with this choice of f^δ, g^δ one may apply the results of Sections 4.1 and 4.2. From (81) we obtain the lower bound in $B_{\frac{\rho}{\delta}} \setminus \overline{B_1}$:

$$\hat{F}_\xi(\hat{v}_\varepsilon, B_{\frac{\rho}{\delta}} \setminus \overline{B_1}) \geq J_\delta + o_\varepsilon(1) \tag{82}$$

with J_δ the energy associate to the minimization problem (Q_δ) (see page 20).

Let v_0 be defined by (44). Since we have $v_\varepsilon \to v_0$ in $H^1(B_1 \setminus \cup \overline{B(\alpha_i, r)})$ and $\hat{U}_\varepsilon \to a$ in $L^2(B_1 \setminus \cup \overline{B(\alpha_i, r)})$, from Proposition 10 and Lemma 4 we obtain

$$\begin{aligned} \hat{F}_\xi(\hat{v}_\varepsilon, B_1 \setminus \cup \overline{B(\alpha_i, r)}) &\geq \frac{1}{2} \int_{B_1 \setminus \cup \overline{B(\alpha_i, r)}} a^2 |\nabla v_0|^2 + o_\varepsilon(1) \\ &\geq \frac{1}{2} \int_{B_1 \setminus \cup \overline{B(\alpha_i, r)}} a^2 |\nabla \theta + \nabla \psi_0|^2 + o_\varepsilon(1) \\ &= K(r) + \mathcal{O}(r |\ln r|) + o_\varepsilon(1), \end{aligned} \tag{83}$$

where $K(r)$ is defined by (R_r) (see page 22).

In order to complete the proof of the lemma, we need to obtain a sharp lower bound in each ball $B(\alpha_i, r)$. Actually we will prove that

$$\hat{F}_\xi(\hat{v}_\varepsilon, B(\alpha_i, r)) \geq b^2 I(\xi/b, r) + o_r(1) + o_\varepsilon(1), \tag{84}$$

with $I(\xi/b, r)$ being defined in (77). The estimate (84) is equivalent to

$$\hat{F}_\xi(\hat{v}_\varepsilon, B(\alpha_i, r)) \geq b^2 I(\xi/b, r + r^2) + o_r(1) + o_\varepsilon(1). \tag{85}$$

Indeed by (78) we have $I(\xi, r + r^2) - I(\xi, r) = o_r(1)$.

We now make use of the construction by Lefter and Rădulescu in [20] and [19]. From Proposition 10, we know that $v_0 = \mathrm{e}^{\imath(\theta_i + \varphi_\star + \psi_i)}$ with $\varphi_\star$, ψ_i harmonic, and therefore smooth in $B(\alpha_i, \eta)$ ($\eta > r$

small). Set $\sigma_i = \varphi_\star + \psi_i$. Without loss of generality, we can assume that $\alpha_i = 0$ and $\sigma_i(0) = 0$. Consequently, $|\sigma_i(x)| \le C|x|$ with C independent of η and $|x| \le \eta$. Let

$$\hat{v}_\varepsilon = \lambda_\varepsilon \mathrm{e}^{\imath(\theta_i + \sigma^i_\varepsilon)} \text{ where } \lambda_\varepsilon := |\hat{v}_\varepsilon|.$$

From Proposition 8 and (54), we obtain that

$$\sigma^i_\varepsilon \to \sigma_i \text{ in } H^1(B_{r+r^2} \setminus \overline{B_r}), \tag{86}$$

$$\lambda_\varepsilon \to 1 \text{ in } H^1(B_{r+r^2} \setminus \overline{B_r}) \text{ and } \frac{1}{\xi^2}\int_{B_{r+r^2}\setminus\overline{B_r}} (1-\lambda_\varepsilon)^2 \to 0. \tag{87}$$

Let

$$\beta_\varepsilon(s\mathrm{e}^{\imath\theta_i}) = \begin{cases} \hat{v}_\varepsilon(s\mathrm{e}^{\imath\theta_i}) & \text{if } s \in [0, r) \\ \left[\dfrac{1-\lambda_\varepsilon}{r^2}(s-r) + \lambda_\varepsilon\right] \exp\left\{\imath\left(\theta_i + \sigma^i_\varepsilon \dfrac{-s+r^2+r}{r^2}\right)\right\} & \text{if } s \in [r, r+r^2] \end{cases}.$$

Clearly, $\beta_\varepsilon \in H^1_{x/|x|}(B_{r+r^2})$. Consequently,

$$b^2 I(\xi/b, r+r^2) \le \hat{F}_\xi(\hat{v}_\varepsilon, B_r) + \hat{F}_\xi(\beta_\varepsilon, B_{r+r^2}\setminus\overline{B_r}) + o_\varepsilon(1).$$

From (87), we easily obtain that

$$\int_{B_{r+r^2}\setminus\overline{B_r}} \left\{ |\nabla|\beta_\varepsilon||^2 + \frac{1}{\xi^2}(1-|\beta_\varepsilon|)^2 \right\} = o_\varepsilon(1).$$

It remains to estimate

$$\int_{B_{r+r^2}\setminus\overline{B_r}} \left|\nabla\left\{\theta_i + \sigma^i_\varepsilon\frac{-s+r^2+r}{r^2}\right\}\right|^2.$$

From (86)

$$\int_{B_{r+r^2}\setminus\overline{B_r}} \left|\nabla\left\{\theta_i + \sigma^i_\varepsilon\frac{-s+r^2+r}{r^2}\right\}\right|^2 = \int_{B_{r+r^2}\setminus\overline{B_r}} \left|\nabla\left\{\theta_i + \sigma_i\frac{-s+r^2+r}{r^2}\right\}\right|^2 + o_\varepsilon(1).$$

Since $|\sigma_i(s\mathrm{e}^{\imath\theta})| \le Cs$, $|\partial_s\sigma_i| \le C$ and $|\partial_{\theta_i}\sigma_i| \le Cs$ we have

$$\begin{aligned} \left|\nabla\left\{\theta_i + \sigma_i\frac{-s+r^2+r}{r^2}\right\}\right|^2 &= \left|\partial_s\sigma_i\frac{-s+r^2+r}{r^2} - \frac{\sigma_i}{r^2}\right|^2 + \frac{1}{r^2}\left|1 + \partial_{\theta_i}\sigma_i\frac{-s+r^2+r}{r^2}\right|^2 \\ &\le C\left[(1+r^{-2}) + r^{-2}\right] = \mathcal{O}(r^{-2}). \end{aligned}$$

Since $|B_{r+r^2}\setminus\overline{B_r}| = \mathcal{O}(r^3)$ we find that

$$\int_{B_{r+r^2}\setminus\overline{B_r}} \left|\nabla\left\{\theta_i + \sigma^i_\varepsilon\frac{-s+r^2+r}{r^2}\right\}\right|^2 = \mathcal{O}(r).$$

It follows that $\hat{F}_\xi(\beta_\varepsilon, B_{r+r^2}\setminus\overline{B_r}) = \mathcal{O}(r) + o_\varepsilon(1)$. Consequently, (85) holds and thus we obtain (84). Combining (82), (83) and (84), together with (71) and (75), we obtain

$$\begin{aligned} \hat{F}_\xi(\hat{v}_\varepsilon, B_{\frac{\rho}{\delta}}) &\ge I_\delta + K(r) + b^2 I(\xi/b, r) + o_\varepsilon(1) + o_r(1) \\ &= \pi d_0^2 \ln\frac{\rho}{\delta} + \pi d_0 b^2 \ln\frac{b}{\xi} + \tilde{W}_0(f_0) + \tilde{W}(\boldsymbol{\alpha}, g_0) + \\ &\qquad + d_0 b^2\gamma + o_\varepsilon(1) + o_r(1). \end{aligned} \tag{88}$$

The conclusion of the Lemma follows by letting $r \to 0$ as $\varepsilon \to 0$. □

4.5 The function g_0 and the points $\{\alpha_1, ..., \alpha_{d_0}\}$ minimize the renormalized energy

In the previous section, we obtained an expansion for the energy $\hat{F}_\xi(\hat{v}_\varepsilon, B_{\frac{\rho}{\delta}})$ of the model problem. To summarize, using (76), (80) and Theorem 3 we get that there are $g_0 = \lim \operatorname{tr}_{\partial B_1} \hat{v}_\varepsilon$ and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_{d_0}) \in \omega^{d_0}$ s.t.

$$\hat{F}_\xi(\hat{v}_\varepsilon, B_{\frac{\rho}{\delta}}) = \pi d_0 b^2 \ln \frac{b}{\xi} + \pi d_0^2 \ln \frac{\rho}{\delta} + \tilde{W}(\boldsymbol{\alpha}, g_0) + \tilde{W}_0(f_0) + d_0 b^2 \gamma + o_\varepsilon(1), \tag{89}$$

with

$$\tilde{W}(\boldsymbol{\alpha}, g_0) = \tilde{W}_1(g_0) + \tilde{W}_2(\boldsymbol{\alpha}, g_0).$$

The goal of this section is to underline an important property of the points $\boldsymbol{\alpha}$, namely, that they minimize the quantity $\inf_{g^0 \in C^\infty(\partial B_1, \mathbb{S}^1)} \tilde{W}(\cdot, g^0)$.

We have the following

Proposition 14. *Let $\boldsymbol{\beta} = (\beta_1, ..., \beta_{d_0}) \in \omega^{d_0}$ be a d_0-tuple of distinct points and let $g^0 \in C^\infty(\partial B_1, \mathbb{S}^1)$ be s.t. $\deg_{\partial B_1}(g^0) = d_0$. Then*

$$\tilde{W}(\boldsymbol{\alpha}, g_0) \leq \tilde{W}(\boldsymbol{\beta}, g^0).$$

Proof. Let $(\boldsymbol{\beta}, g^0)$ be as in Proposition 14. Using the test function given by (79), we obtain that for all $\varepsilon > 0$ and $r > 0$ (small) there is $w_\varepsilon \in H^1_{\hat{f}_\varepsilon}(B_{\frac{\rho}{\delta}}, \mathbb{C})$ s.t.

$$\hat{F}_\xi(w_\varepsilon) = \pi d_0 b^2 \ln \frac{b}{\xi} + \pi d_0^2 \ln \frac{\rho}{\delta} + \tilde{W}(\boldsymbol{\beta}, g^0) + \tilde{W}_0(f_0) + d_0 b^2 \gamma + h^1_\varepsilon + h^2_r$$

here $h^1_\varepsilon = o_\varepsilon(1)$ and $h^2_r = \mathcal{O}(r)$.

On the other hand, taking into account the minimality of $\hat{v}_\varepsilon$ and (89) we have

$$\tilde{W}(\boldsymbol{\beta}, g^0) \geq \tilde{W}(\boldsymbol{\alpha}, g_0) + o_\varepsilon(1) + h^2_r.$$

The previous estimate implies (letting $\varepsilon \to 0$ and $r \to 0$) that $\tilde{W}(\boldsymbol{\beta}, g^0) \geq \tilde{W}(\boldsymbol{\alpha}, g_0)$ which completes the proof. □

Thus, for $\boldsymbol{\beta} = (\beta_1, ..., \beta_{d_0}) \in \omega^{d_0}$ we define

$$\tilde{W}(\boldsymbol{\beta}) = \inf_{\substack{\tilde{g} \in C^\infty(\partial B_1, \mathbb{S}^1) \\ \deg_{\partial B_1}(\tilde{g}) = d_0}} \tilde{W}(\boldsymbol{\beta}, \tilde{g}) = \inf_{\substack{\tilde{g} \in C^\infty(\partial B_1, \mathbb{S}^1) \\ \deg_{\partial B_1}(\tilde{g}) = d_0}} \tilde{W}_1(\tilde{g}) + \tilde{W}_2(\boldsymbol{\beta}, \tilde{g}) \tag{90}$$

with $\tilde{W}_1$ and $\tilde{W}_2$ given by (72) and (75) respectively. It follows that for $\boldsymbol{\alpha}$ given by Theorem 3 and $g_0 = \operatorname{tr}_{\partial B_1} v_0$:

$$\tilde{W}(\boldsymbol{\alpha}) = \tilde{W}(\boldsymbol{\alpha}, g_0) \leq \tilde{W}(\boldsymbol{\beta}) \text{ for all } \boldsymbol{\beta} = (\beta_1, ..., \beta_{d_0}) \in \omega^{d_0}.$$

5 Proofs of Theorems 1 and 2

In this section v_ε is a minimizer of F_ε in $H^1_g(\Omega, \mathbb{C})$. We split the proofs of Theorem 1 and 2 in three steps:

- (Section 5.1) Using estimates on $|v_\varepsilon|$, we first localize the vorticity to the neighborhoods of selected inclusions. Then we find two separate energy expansions in two sub-domains of Ω: away from the selected inclusions and around them.
- (Section 5.2) We study the asymptotic behavior of v_ε. We prove that, for small ε, v_ε has exactly d zeros of degree 1.
- (Section 5.3) We give an expansion of $F_\varepsilon(v_\varepsilon)$ up to $o_\varepsilon(1)$ terms and relate the choice of the inclusions with vortices to the renormalized energy of Bethuel, Brezis and Hélein.

5.1 Locating bad inclusions

The following result gives a uniform bound on the modulus of minimizers away from the inclusions.

Lemma 7. *There exists $C>0$ s.t. for small ε we have*

1. $|v_\varepsilon| \geq 1 - C|\ln\varepsilon|^{-1/3}$ *in* $\Omega \setminus \cup_{i=1}^M \overline{B(a_i,\delta)}$,
2. *there are at most d points $a_{i_1},...,a_{i_{d'}}$ ($1 \leq d' = d'_\varepsilon \leq d$) s.t.* $\{|v_\varepsilon| < 1 - C|\ln\varepsilon|^{-1/3}\} \subset \cup_{k=1}^{d'} B(a_{i_k},\delta)$.

Proof. Using Lemma 1 with $\chi = |\ln\varepsilon|^{-1/3}$, we obtain that there exist $C, C_1 > 0$ s.t. for $\varepsilon > 0$ small,

$$\text{if } F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) < |\ln\varepsilon|^{\frac{1}{3}} - C_1 \text{ then } |v_\varepsilon| \geq 1 - C\chi \text{ in } B(x,\varepsilon^{1/2}).$$

We prove 1. by contradiction. Assume that, up to a subsequence, there is $x_\varepsilon \in \Omega \setminus \cup_{i=1}^M \overline{B(a_i,\delta)}$, s.t. $|v_\varepsilon(x_\varepsilon)| < 1 - C|\ln\varepsilon|^{-1/3}$ with C given by Lemma 1. From Lemma 1 and Proposition 2

$$\frac{1}{2}\int_{B(x_\varepsilon,\varepsilon^{1/4})} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \right\} \geq |\ln\varepsilon|^{1/3} - \mathcal{O}(1). \tag{91}$$

Fix a bounded, simply connected domain Ω' such that $\overline{\Omega} \subset \Omega'$, and extend v_ε by a fixed smooth $\mathbb{S}^1$-valued map v in $\Omega' \setminus \overline{\Omega}$, s.t. $v = g$ on $\partial\Omega$.

In view of (11) for Case I or (12) for Case II, there exists $\tilde{C} > 0$ s.t. for small ε

$$\frac{1}{2}\int_{\Omega'} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq \tilde{C}|\ln\varepsilon|.$$

Therefore, the map v_ε in Ω' satisfies the condition of Theorem 4.1 [27]. This theorem guarantees that

- there exists $\mathcal{B}^\varepsilon = \{B_j^\varepsilon\}$, a finite disjoint covering of the set

$$\{x \in \Omega' \,|\, \mathrm{dist}(x,\partial\Omega') > \varepsilon/b \text{ and } |v_\varepsilon(x)| < 1 - (\varepsilon/b)^{1/8}\},$$

- such that $\mathrm{rad}(\mathcal{B}^\varepsilon) := \sum_j \mathrm{rad}(B_j^\varepsilon) \leq 10^{-2}\cdot \mathrm{dist}(\omega, \partial B(0,1))\cdot\delta$,
- and, denoting $d_j = |\deg_{\partial B_j}(v_\varepsilon)|$ if $B_j^\varepsilon \subset \{\mathrm{dist}(x,\partial\Omega') > \varepsilon/b\}$ and $d_j = 0$ otherwise, we have

$$\begin{aligned} \frac{1}{2}\int_{\cup B_j^\varepsilon} |\nabla v_\varepsilon|^2 + \frac{b^2}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 &\geq \pi\sum_j d_j \ln\frac{\delta}{\varepsilon} - C \\ &= \pi\sum_j d_j|\ln\xi| - C, \end{aligned} \tag{92}$$

  with C independent of ε.

Note that since $|v_\varepsilon| \equiv 1$ in $\Omega' \setminus \overline{\Omega}$, if $d_j \neq 0$ then $B_j^\varepsilon \subset \{\mathrm{dist}(x,\partial\Omega') > \varepsilon/b\}$. Consequently, we have $d_j = |\deg_{\partial B_j^\varepsilon}(v_\varepsilon)|$.

Assertion 1. follows as in the proof of Proposition 5 (use (91), (92) instead of (20) and (22)).

The proof of Assertion *2.* of Lemma 7 goes along the same lines. □

We next obtain the following lower bounds for the energy.

Lemma 8. *For $k \in \{1, ..., d'\}$, we denote $d_k = d_k^\varepsilon = \deg_{\partial B(a_{i_k},\delta)}(v_\varepsilon)$. There exist $C, \eta_0 > 0$ s.t. for small ε and $\rho \in [2\delta, \eta_0]$ we have*

$$\frac{1}{2}\int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} |\nabla v_\varepsilon|^2 \geq \pi \sum_{k=1}^d d_k^2 |\ln \rho| - C \tag{93}$$

and

$$F_\varepsilon(v_\varepsilon, B(a_{i_k}, 2\delta)) \geq \pi |d_k| b^2 |\ln \xi| - C. \tag{94}$$

Proof. Let $\eta_0 = 10^{-2} \min_i \{\mathrm{dist}(a_i, \partial\Omega), \min_{j\neq i} |a_i - a_j|\}$ and $0 < \rho < \eta_0$.

We prove (93). By Lemma 7, $|v_\varepsilon| \geq 1/2$ in $\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}$, therefore, $w_\varepsilon = \frac{v_\varepsilon}{|v_\varepsilon|}$ is well-defined in this domain. From direct computations in $B(a_{i_k}, \eta_0) \setminus \overline{B(a_{i_k}, \rho)}$ we have

$$\frac{1}{2}\int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} |\nabla w_\varepsilon|^2 \geq \pi \sum_{i=1}^d d_i^2 \ln \frac{\eta_0}{\rho}. \tag{95}$$

We claim that the bound (93) holds with $C = |\ln \eta_0| + 1$. Argue by contradiction: assume that up to a subsequence we have:

$$\frac{1}{2}\int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} |\nabla v_\varepsilon|^2 \leq \pi \sum_{i=1}^d d_i^2 \ln \frac{\eta_0}{\rho} - 1. \tag{96}$$

On the other hand, we have

$$|\nabla v_\varepsilon|^2 = |v_\varepsilon|^2 |\nabla w_\varepsilon|^2 + |\nabla |v_\varepsilon||^2$$

and therefore

$$\int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} |\nabla v_\varepsilon|^2 \geq \int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} |\nabla w_\varepsilon|^2 - (1 - |v_\varepsilon|^2)|\nabla w_\varepsilon|^2. \tag{97}$$

Using the fact that $|v_\varepsilon| \geq \frac{1}{2}$ in $\Omega \setminus \cup_{k=1}^d \overline{B(a_{i_k}, \rho)}$ we see that $|\nabla w_\varepsilon| \leq 2|\nabla v_\varepsilon|$. Therefore, by (96), (H) and Lemma 7 we estimate the last term in (97):

$$\int_{\Omega\setminus\cup_{k=1}^d \overline{B(a_{i_k},\rho)}} (1 - |v_\varepsilon|^2)|\nabla w_\varepsilon|^2 \leq C |\ln \varepsilon|^{-\frac{1}{3}} \int_\Omega |\nabla v_\varepsilon|^2 \leq C \frac{|\ln \rho|}{|\ln \varepsilon|^{\frac{1}{3}}} \to 0. \tag{98}$$

By combining (95), (97) and (98), we see that (96) cannot hold for small ε; this implies (93).
We now prove (94). Performing the rescaling $\hat{x} = \frac{x - a_{i_k}}{\delta}$, we obtain

$$F_\varepsilon(v, B(a_{i_k}, 2\delta)) = \hat{F}_\xi(\hat{v}, B(0, 2)) = \frac{1}{2}\int_{B(0,2)} \left\{ \hat{U}_\varepsilon^2 |\nabla \hat{v}|^2 + \frac{1}{2\xi^2} \hat{U}_\varepsilon^4 (1 - |\hat{v}|^2)^2 \right\} \mathrm{d}\hat{x},$$

where, as in the model problem we set $\hat{v}(\hat{x}) = v(\delta \hat{x})$ and $\xi = \dfrac{\varepsilon}{\delta}$.

By Theorem 4.1 [27], for $r = 10^{-2}$ there are $C > 0$ and a finite covering by disjoint balls $B_1, ..., B_N$ (with the sum of radii at most r) of $\{\hat{x} \in B(0, 2 - \xi/b) \,|\, 1 - |\hat{v}_\varepsilon(\hat{x})| \geq (\xi/b)^{1/8}\}$ and

$$\frac{1}{2}\int_{\cup_j B_j} \left\{ |\nabla \hat{v}_\varepsilon|^2 + \frac{b^2}{2\xi^2}(1 - |\hat{v}_\varepsilon|^2)^2 \right\} \geq \pi D_k |\ln \xi| - C, \tag{99}$$

$D_k = \sum_j |m_j|$ and

$$m_j = \begin{cases} \deg_{\partial B_j}(\hat{v}_\varepsilon) & \text{if } \mathrm{dist}(B_j, \partial B(0,2)) \geq \xi/b \\ 0 & \text{otherwise} \end{cases}.$$

Since, by Lemma 7, $|\hat{v}_\varepsilon| \geq 1/2$ in $B(0, 2) \setminus \overline{B(0, 1)}$, $D_k \geq d_k$, and (94) follows from (99) and the estimate $U_\varepsilon \geq b$.

□

Corollary 5. *Assume that $M \geq d$. Then $d' = d$ and $d_k = 1$ for each k.*

Corollary 6. *Assume that $M < d$. Then $d' = M$ and $d_k \in \left\{ \left[\dfrac{d}{M}\right], \left[\dfrac{d}{M}\right] + 1 \right\}$ for each k.*

Proof of Corollaries 5 and 6. By combining (93) and (94) we obtain the lower bound for F_ε in Ω:

$$F_\varepsilon(v_\varepsilon) + C_1 \geq \pi \sum_{i=1}^{M} \left\{ \deg_{\partial B(a_i,\delta)}(v_\varepsilon)^2 |\ln \delta| + b^2 |\deg_{\partial B(a_i,\delta)}(v_\varepsilon)| |\ln \xi| \right\}. \tag{100}$$

The conclusions of the above corollaries are obtained by solving the discrete minimization problem of the RHS of (100). $\square$

As a direct consequence of Proposition 3 and Lemma 8, we have

Corollary 7. *There is $C > 0$ independent of ε s.t. for $1 > \rho > 2\delta$ we have*

$$\begin{aligned}
\frac{1}{2} \int_{\Omega \setminus \cup_{k=1}^{d} \overline{B(a_{i_k},\rho)}} |\nabla v_\varepsilon|^2 \, \mathrm{d}x &= \pi \sum_{k=1}^{d'} d_k^2 |\ln \rho| + \mathcal{O}(1) \\
&= \begin{cases} \pi d |\ln \rho| + \mathcal{O}(1) & \textit{in Case I} \\ \pi \displaystyle\min_{\substack{\tilde{d}_1,\dots,\tilde{d}_M \in \mathbb{Z} \\ \tilde{d}_1+\dots+\tilde{d}_M = d}} \sum_{i=1}^{M} \tilde{d}_i^2 |\ln \rho| + \mathcal{O}(1) & \textit{in Case II} \end{cases}
\end{aligned}$$

.

5.2 Existence of the limiting solution

We now return to the proof of Theorems 1 and 2.

Recall that $\{i_1^\varepsilon, \dots, i_{d'}^\varepsilon\}$ is a set of distinct elements of $\{1, \dots, M\}$. We choose ε small enough so that i_j's are independent of ε, thus we may simply denote this set by $\{i_1, \dots, i_{d'}\}$. In Case I, we have $d' = d$ and we may assume that $\{i_1, \dots, i_{d'}\} = \{1, \dots, d\}$. In Case II, we have $d' = M$.

Lemma 7 and Corollary 7 imply that for an appropriate extraction $\varepsilon = \varepsilon_n \downarrow 0$ and for a compact $K \subset \Omega \setminus \{a_{i_1}, \dots, a_{i_{d'}}\}$, there is $C_K > 0$ s.t. for small ε we have

$$F_\varepsilon(v_\varepsilon, K) \leq C_K$$

and

$$|v_\varepsilon(x)| \geq 1 - C|\ln \varepsilon|^{-1/3} \text{ for all } x \in K.$$

Therefore, when $\varepsilon \to 0$, up to a subsequence, there exists $v^* \in H^1(\Omega \setminus \{a_{i_1}, \dots, a_{i_{d'}}\}, \mathbb{S}^1)$ s.t. $v_\varepsilon \rightharpoonup v^* \in H^1_{\text{loc}}(\Omega \setminus \{a_{i_1}, \dots, a_{i_{d'}}\})$.

We now fix such sequence and a compact $K \subset \Omega \setminus \{a_{i_1}, \dots, a_{i_{d'}}\}$. If $K \subset \Omega \setminus \{a_i, 1 \leq i \leq M\}$, then we have $K \cap \omega_\delta = \emptyset$ for small ε. By exactly the same argument as in Proposition 7 we deduce that v_ε is bounded in $C^k(K)$ for all $k \geq 0$ and $1 - |v_\varepsilon|^2 \leq C_K \varepsilon^2$ in K.

Consequently, up to subsequence we have for a compact set $K \subset \Omega \setminus \{a_1, \dots, a_M\}$

$$v_\varepsilon \to v^* \text{ in } C^k(K) \text{ and } 1 - |v_\varepsilon|^2 \leq C_K \varepsilon^2. \tag{101}$$

Now, assume that K is s.t. $K \subset \Omega \setminus \{a_{i_1}, \dots, a_{i_{d'}}\}$ but $K \cap \omega_\delta \neq \emptyset$ (then we are in Case I). Without loss of generality, assume $K = \overline{B(a_{k_0}, R)}$, where $a_{k_0} \in \{a_{d+1}, \dots, a_M\}$ and $R > 0$ is sufficiently small in order to have $K \cap \{a_1, \dots, a_M\} = \{a_{k_0}\}$.

Let $h_\varepsilon := \mathrm{tr}_{\partial K} v_\varepsilon$. Since $\partial K \subset \Omega \setminus \{a_1, ..., a_M\}$, we have $h_\varepsilon \to h_0$ in $C^\infty(\partial K)$ (possibly after passing to a subsequence). Since $\deg(h_\varepsilon, \partial K) = 0$ we have $\deg(h_0, \partial K) = 0$ and consequently there is some $\varphi_0 \in C^\infty(\partial K, \mathbb{R})$ s.t. $h_0 = e^{i\varphi_0}$.

Let $\tilde{v}$ be a minimizer of $\displaystyle\int_K |\nabla v|^2$ in the class $H^1_{h_0}(K, \mathbb{S}^1)$. Clearly,

$$\int_K |\nabla \tilde{v}|^2 \le \int_K |\nabla v^*|^2.$$

On the other hand, since $U_\varepsilon \le 1$, we may construct (in the spirit of [10]) a test function and find that (see formula (93) in [10])

$$F_\varepsilon(v_\varepsilon, K) \le \frac{1}{2}\int_K |\nabla \psi_\varepsilon|^2 + C\varepsilon, \tag{102}$$

where ψ_ε is the solution of

$$\begin{cases} \Delta\psi_\varepsilon = 0 & \text{in } K \\ \psi_\varepsilon = \varphi_\varepsilon & \text{on } \partial K \end{cases}.$$

Here, φ_ε is defined by

$$e^{i\varphi_\varepsilon} = \frac{h_\varepsilon}{|h_\varepsilon|} \text{ on } \partial K.$$

As $\varepsilon \to 0$, we have

$$\psi_\varepsilon \to \psi_0 \text{ strongly in } H^1(K), \text{ where } \begin{cases} \Delta\psi_0 = 0 & \text{in } K \\ \psi_0 = \varphi_0 & \text{on } \partial K \end{cases}. \tag{103}$$

From the fact that $v_\varepsilon \rightharpoonup v_*$ in $L^2(K)$, $U_\varepsilon \to 1$ in $L^2(K)$ and $|U_\varepsilon| \le 1$ we have $U_\varepsilon^2 \nabla v_\varepsilon \rightharpoonup v^*$ in $L^2(K)$. Consequently, we obtain

$$\frac{1}{2}\int_K |\nabla v^*|^2 \;\le\; \liminf_{\varepsilon\to 0} \frac{1}{2}\int_K U_\varepsilon^2 |\nabla v_\varepsilon|^2 \le \liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon, K). \tag{104}$$

Combining (102), (103) and (104) we deduce that

$$\int_K |\nabla v^*|^2 \le \int_K |\nabla \psi_0|^2 = \int_K |\nabla \tilde{v}|^2.$$

It follows that v^* minimizes the Dirichlet functional in

$$H^1_{h_0}(K, \mathbb{S}^1) := \{v \in H^1(K, \mathbb{S}^1), v = h_0 \text{ on } \partial K\}.$$

We find that hence $\tilde{v} = v^*$ in K. By a classic result of Morrey [24] (see also [10]), v^* satisfies (5). Moreover, as follows from weak lower semicontinuity of Dirichlet integral, (102), (103) and (104)

$$\frac{1}{2}\int_K |\nabla v^*|^2 \le \liminf_{\varepsilon\to 0} \frac{1}{2}\int_K |\nabla v_\varepsilon|^2 \le \limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon, K) \le \frac{1}{2}\int_K |\nabla v^*|^2.$$

Therefore,

$$v_\varepsilon \text{ converges to } v^* \text{ strongly in } H^1(K). \tag{105}$$

From (101) and (105) we obtain that $v_\varepsilon \to v^*$ in $H^1_{\mathrm{loc}}(\Omega \setminus \{a_1, ..., a_{d'}\})$. The convergence up to $\partial\Omega$ will be established in the next section.

In order to prove Assertion *3.* of Theorem 1 and Assertion *2.* of Theorem 2, note that, for small $\rho > 0$, the estimate (101) implies that $f_\varepsilon := \mathrm{tr}_{\partial B(a_{i_k}, \rho)} v_\varepsilon$ satisfies the conditions (13) and (14) of Theorem 3. This gives us *3.* of Theorem 1 and *2.* of Theorem 2.

Assertion *3.* of Theorem 2 is is a consequence of Corollary 6.

5.3 The macroscopic position of vortices minimizes the Bethuel-Brezis-Hélein renormalized energy

Let us recall briefly the concept of the renormalized energy $W_g((b_1, d_1), ..., (b_k, d_k))$ with

$$\begin{cases} g \in C^\infty(\partial\Omega, \mathbb{S}^1) \text{ s.t. } \deg_{\partial\Omega}(g) = d \\ b_1, ..., b_k \in \Omega,\ b_i \neq b_j \text{ for } i \neq j \\ d_i \in \mathbb{Z} \text{ and } \sum_i d_i = d \end{cases}.$$

For small $\rho > 0$, consider $\Omega_\rho = \Omega \setminus \cup_i \overline{B(b_i, \rho)}$ and the minimization problem

$$I_\rho((b_1, ..., b_k), (d_1, ..., d_k)) = \inf_{\substack{w \in H^1(\Omega_\rho, \mathbb{S}^1) \text{ s.t.} \\ w = g \text{ on } \partial\Omega \\ w(b_i + \rho e^{\imath\theta}) = \alpha_i e^{\imath d_i \theta},\, \alpha_i \in \mathbb{S}^1}} \frac{1}{2} \int_{\Omega_\rho} |\nabla w|^2.$$

Such problem is studied in detail in [11] (Chapter 1). In particular Bethuel, Brezis and Hélein proved that for small ρ, we have

$$I_\rho((b_1, ..., b_k), (d_1, ..., d_k)) = \pi d |\ln \rho| + W_g((b_1, d_1), ..., (b_k, d_k)) + o_\rho(1).$$

This equality plays an important role in the study done in [11]. In the minimization problem of the classical Ginzburg-Landau functional

$$\frac{1}{2} \int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right\},\ u \in H^1_g,$$

the vortices (with their degrees) of a minimizer tend to form (up to a subsequence) a minimal configuration for W_g.

We prove in this section that the (macroscopic) location of the vorticity of minimizers of F_ε is related to the minimization problem of $W_g((b_1, ..., b_k), (d_1, ..., d_k))$ with $b_1, ..., b_k \in \{a_1, ..., a_M\}$.

We present here the argument for Case I (Theorem 1). The argument in Case II is analogous.

The proof of Assertion *4.* relies on two lemmas, providing sharp upper and lower bounds.

Lemma 9. *There exists $\rho_0 > 0$ s.t., for every $\rho < \rho_0$ and every $\varepsilon > 0$, we have*

$$F_\varepsilon(v_\varepsilon) \leq \pi d |\ln \rho| + dJ(\varepsilon, \rho) + W_g((a_{i_1}, 1), ..., (a_{i_d}, 1)) + o_\rho(1), \tag{106}$$

where $J(\varepsilon, \rho) = \inf_{u \in H^1_{g_\rho}(B_\rho(0))} F_\varepsilon(u)$ with $g_\rho = \frac{z}{\rho}$ on $\partial B(0, \rho)$.

Proof. The proof, via construction of a test function, is the same as proof of Lemma VIII.1 in [11]. □

Lemma 10. *Let $\rho > 0$, $\rho < \rho_0$. Then for small ε we have*

$$F_\varepsilon(v_\varepsilon) \geq \pi d |\ln \rho| + dJ(\varepsilon, \rho) + W_g((a_{i_1}, 1), ..., (a_{i_d}, 1)) + o_\rho(1). \tag{107}$$

Proof. Split the domain Ω into two sub-domains: $\Omega \setminus \cup_i \overline{B(a_{k_i}, \rho)}$ and $\cup_i B(a_{k_i}, \rho)$. We start with the lower bound in the first sub-domain. By the previous estimate, v_ε weakly converges to v^* in $H^1(\Omega \setminus \cup_i \overline{B(a_{k_i}, \rho)})$. This implies that

$$\liminf \frac{1}{2} \int_{\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)}} U_\varepsilon^2 |\nabla v_\varepsilon|^2 \geq \frac{1}{2} \int_{\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)}} |\nabla v^*|^2.$$

Here, we used the fact that, since $U_\varepsilon \to 1$ in $L^2(\Omega)$, $|U_\varepsilon| \leq 1$ and $\nabla v_\varepsilon \rightharpoonup \nabla v^*$ in $L^2(\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)})$, we have $U_\varepsilon \nabla v_\varepsilon \rightharpoonup \nabla v^*$ in $L^2(\Omega \setminus \cup_k \overline{B(a_{i_k}, \rho)})$.

Thus we deduce that, for small ε,

$$\frac{1}{2}\int_{\Omega\setminus\cup_k\overline{B(a_{i_k},\rho)}} U_\varepsilon^2|\nabla v_\varepsilon|^2 \geq \frac{1}{2}\int_{\Omega\setminus\cup_k\overline{B(a_{i_k},\rho)}} |\nabla v^*|^2 - \rho^2. \tag{108}$$

On the other hand, as proved in [11],

$$\frac{1}{2}\int_{\Omega\setminus\cup_k\overline{B(a_{i_k},\rho)}} |\nabla v^*|^2 \geq \pi d\ln\frac{1}{\rho} + W_g((a_{i_1},1),...,(a_{i_d},1)) + o_\rho(1). \tag{109}$$

Thus, combining (108), (109) and using Proposition 2, for ε sufficiently small, we have

$$F_\varepsilon(v_\varepsilon,\Omega\setminus\cup_k\overline{B(a_{i_k},\rho)}) \geq \pi d\ln\frac{1}{\rho} + W_g((a_{i_1},1),...,(a_{i_d},1)) + o_\rho(1). \tag{110}$$

By Theorem 4 and Corollary 5 we have the following energy expansion:

$$F_\varepsilon(v_\varepsilon,B(a_{i_k},\rho)) = \pi\ln\rho + \pi b^2|\ln\varepsilon| + \pi(1-b^2)|\ln\delta| + \tilde{W}(\boldsymbol{\alpha}) + \tilde{W}_0(f_0) + b^2\gamma + o_\varepsilon(1). \tag{111}$$

Similarly, applying Theorem 4 to $J(\varepsilon,\rho)$ we obtain

$$J(\varepsilon,\rho) = \pi\ln\rho + \pi b^2|\ln\varepsilon| + \pi(1-b^2)|\ln\delta| + \tilde{W}(\boldsymbol{\alpha}) + \tilde{W}_0(z/|z|) + b^2\gamma + o_\varepsilon(1). \tag{112}$$

Here, the local renormalized energy $\tilde{W}(\boldsymbol{\alpha})$ is given by (90) and is the same in (111) and (112).

From (73), $\tilde{W}_0(f_0)\geq 0$ while $\tilde{W}_0(\frac{z}{|z|})=0$. Consequently, we have $F_\varepsilon(v_\varepsilon,B(a_{i_k},\rho))-J(\varepsilon,\rho)\geq o_\varepsilon(1)$. Hence $\forall\rho>0$ there exists $\varepsilon_\rho>0$ s.t. for $\varepsilon<\varepsilon_\rho$ we have

$$F_\varepsilon(v_\varepsilon,B(a_{i_k},\rho)) \geq J(\varepsilon,\rho)-\rho^2$$

and thus

$$F_\varepsilon(v_\varepsilon,\cup_k B(a_{i_k},\rho)) \geq dJ(\varepsilon,\rho)-d\rho^2, \tag{113}$$

which gives the lower bound in the second sub-domain. From (110) and (113) the bound (107) follows. $\square$

Combining Lemma 9 and Lemma 10, we see that the points $\{a_{i_k}, 1\leq k\leq d\}$ minimize W_g among $a_1,...,a_M$. The expansion (6) follows from (106), (107) and (112).

We next turn to convergence of v_ε up to the boundary. It suffices to prove the H^1-convergence of v_ε in $\Omega_\rho=\Omega\setminus\cup_m\overline{B(a_{i_m},\rho)}$ (for small $\rho>0$). We argue by contradiction and we assume that there are some $\rho_1>0$ and $\eta>0$ s.t.

$$\liminf\frac{1}{2}\int_{\Omega_{\rho_1}}|\nabla v_\varepsilon|^2 \geq \frac{1}{2}\int_{\Omega_{\rho_1}}|\nabla v^*|^2+\eta. \tag{114}$$

Note that for all $\rho\leq\rho_1$, (114) still holds in Ω_ρ.

If, in the proof of Lemma 10, we replace (108) by (114) (with ρ_1 replaced by ρ), then we obtain for small ρ a contradiction with Lemma 9. The proof of Theorem 1 is complete. The last assertion of Theorem 2 is obtained along the same lines.

A Proof of Proposition 2

Let $x_0\in V_R$ be s.t. $B_R=B(x_0,R)\subset\Omega\setminus\overline{\omega_\delta}$ and assume that $x_0=0$.

We follow the proof of Lemma 2 in [10].

In B_R, $\eta = 1 - U_\varepsilon$ satisfies

$$\begin{cases} -\varepsilon^2\Delta\eta + t\eta = -\eta(\eta^2 - 3\eta + 2 - t) & \text{in } B_R \\ \eta \leq 1 & \text{on } \partial B_R \end{cases},$$

here, t will be chosed later.

Since $\eta \in (0, 1-b)$, if we take $t = b(1+b)$, then we have

$$-\varepsilon^2\Delta\eta + t\eta \leq 0 \text{ in } B_R.$$

On the other hand, the function $w(x) = e^{\gamma(|x|^2 - R^2)}$ satisfies

$$\begin{cases} -\varepsilon^2\Delta w + tw = \left[-4\varepsilon^2\gamma(1+\gamma|x|^2) + t\right] w & \text{in } B_R \\ w = 1 & \text{on } \partial B_R \end{cases}.$$

A simple computation gives that

$$\begin{cases} -\varepsilon^2\Delta w + tw \geq 0 \text{ in } B_R \\ \gamma > 0 \end{cases} \Leftrightarrow 0 < \gamma \leq \frac{-\varepsilon + \sqrt{\varepsilon^2 + tR^2}}{2R^2\varepsilon}.$$

Take

$$\gamma = \frac{-\varepsilon + \sqrt{\varepsilon^2 + tR^2}}{2R^2\varepsilon} > 0.$$

Setting $v = \eta - w$, we have

$$\begin{cases} -\varepsilon^2\Delta v + tv \leq 0 & \text{in } B_R \\ v \leq 0 & \text{on } \partial B_R. \end{cases}$$

By the maximum principle, we have $v \leq 0$ in B_R. Therefore,

$$\eta(0) \leq \exp\left\{-\frac{-\varepsilon + \sqrt{\varepsilon^2 + tR^2}}{2\varepsilon}\right\} \leq Ce^{-\frac{\sqrt{t}R}{4\varepsilon}}.$$

Consequently, (9) holds in $\{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) \geq R, \mathrm{dist}(x, \omega_\delta) > R\}$. The estimate close to the $\partial\Omega$ is a direct consequence of $0 \leq U_\varepsilon \leq 1$, (9) holds in $\{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) \geq R, \mathrm{dist}(x, \omega_\delta) > R\}$ and the equation $-\Delta U_\varepsilon = \dfrac{1}{\varepsilon^2}U_\varepsilon(1 - |U_\varepsilon|^2)$ in $\{x \in \Omega \,|\, \mathrm{dist}(x, \omega_\delta) > R\}$. Using a similar argument, we establish (9) in the case $V_R \cap \omega_\delta$. The proof of (9) is complete.

In order to prove (10), note that in $W_R := \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\omega_\delta) \geq R, \mathrm{dist}(x, \partial\Omega) \geq R\}$ the function $\eta = a_\varepsilon - U_\varepsilon$ satisfies $\Delta\eta = \frac{U_\varepsilon}{\varepsilon^2}(a_\varepsilon^2 - U_\varepsilon^2)$. Thus, applying Lemma A.1 [10] to η in conjunction with (9) and the fact that $R \geq \varepsilon$, we obtain

$$|\nabla\eta| \leq \frac{C_1 e^{-\frac{cR}{\varepsilon}}}{\varepsilon} \text{ in } W_R.$$

Thus (10) holds far away from $\partial\Omega$ and the inclusions.

We next prove that the bound (10) holds near $\partial\Omega$.

Indeed, fix a smooth compact $K \subset \Omega$ s.t. for small δ we have $\omega_\delta \subset K$. Clearly, by (9), $0 \leq \eta_K := \mathrm{tr}_{\partial K}\eta \leq Ce^{-\frac{cR}{\varepsilon}}$. In $\Omega \setminus K$, η satisfies

$$\begin{cases} \Delta\eta = \frac{1}{\varepsilon^2}U(1+U)\eta & \text{in } \Omega \setminus K \\ \eta = 0 & \text{on } \partial\Omega \\ \eta = \eta_K & \text{on } \partial K \end{cases}.$$

Let $\eta = \eta_1 + \eta_2$ be s.t. η_1 solves

$$\begin{cases} \Delta\eta_1 = \frac{1}{\varepsilon^2} U(1+U)\eta & \text{in } \Omega \setminus K \\ \eta_1 = 0 & \text{on } \partial\Omega \cup \partial K \end{cases}$$

and η_2 satisfies

$$\begin{cases} \Delta\eta_2 = 0 & \text{in } \Omega \setminus K \\ \eta_2 = 0 & \text{on } \partial\Omega \\ \eta_2 = \eta_K & \text{on } \partial K \end{cases}.$$

Note that $\|\eta_2\|_{L^\infty} \le Ce^{-\frac{cR}{\varepsilon}}$ and thus $\|\eta_1\|_{L^\infty} \le Ce^{-\frac{cR}{\varepsilon}}$.

Lemma A.2 in [10] implies the existence of a constant $C_{\Omega\setminus K} > 0$ s.t.

$$|\nabla\eta_1| \le \frac{C_{\Omega\setminus K} e^{-\frac{cR}{\varepsilon}}}{\varepsilon} \text{ in } \Omega \setminus K.$$

In order to estimate $\nabla\eta_2$ near $\partial\Omega$, we express η_2 in terms of Green's function $G(x,y)$ in $\Omega \setminus K$: function, *i.e.*

$$\eta_2(x) = -\int_{\partial K} \eta_K(y) \frac{\partial G}{\partial \nu}(x,y) dS(y). \tag{115}$$

It follows from (115) and (9) that $|\nabla\eta_2| \le C_0 e^{-\frac{cR}{\varepsilon}}$ away from ∂K. The estimate (10) is proved.

B Proof of Proposition 3

This appendix is devoted to the proof of Proposition 3.

We prove the first assertion: when $M \ge d$ we have

$$\inf_{v \in H^1_g(\Omega)} F_\varepsilon(v,\Omega) \le \pi d b^2 |\ln\xi| + \pi d |\ln\delta| + \mathcal{O}(1).$$

Fix first d distinct points-centers of inclusions $a_1, ..., a_d$. Let $\rho_0 := 10^{-2} \cdot \min(\text{dist}(a_i, \partial\Omega), \min_{i\neq j} |a_i - a_j| > 0)$. Consider $\tilde{v}$ to be a smooth fixed function in $\Omega \setminus \cup_{i=1}^d \overline{B(a_i,\rho_0)}$, such that $|\tilde{v}| = 1$ in $\Omega \setminus \cup_{i=1}^d \overline{B(a_i,\rho_0)}$ and

$$\begin{cases} \tilde{v} = g & \text{on } \partial\Omega \\ \tilde{v}(x) = \frac{x-a_i}{|x-a_i|} & \text{on } \partial B(a_i,\rho_0) \end{cases}.$$

Such a function clearly exists since the compatibility condition $\deg_{\partial\Omega}(g) = \sum_{i=1}^d \deg_{\partial B(a_i,\rho_0)}(\tilde{v})$ is satisfied. Let $c_0 = 10^{-2} \cdot \text{dist}(0, \partial\omega)$. For every $1 \le i \le M$, consider a disc $B(a_i, c_0\delta) \subset \omega^i_\delta$. By the choice of c_0, we have $\text{dist}(\partial\omega_\delta, B(a_i,c_0\delta)) \ge c_0\delta$. Therefore, using Proposition 2

$$U_\varepsilon^2 - b^2 \le Ce^{-\frac{c\delta}{\varepsilon}} \text{ in } B(a_i, c_0\delta). \tag{116}$$

Consider the test function v_0^ε defined as

$$v_0^\varepsilon(x) = \begin{cases} \tilde{v}(x) & \text{for } x \in \Omega \setminus \cup_i \overline{B(a_i,\rho_0)} \\ \dfrac{x-a_i}{|x-a_i|} & \text{for } x \in B(a_i,\rho_0) \setminus \overline{B(a_i,\varepsilon)} \\ \dfrac{x-a_i}{\varepsilon} & \text{for } x \in B(a_i,\varepsilon) \end{cases}.$$

Using (116) and (H) we have

$$\begin{aligned} \inf_{v \in H^1_g(\Omega)} F_\varepsilon(v,\Omega) &\le F_\varepsilon(v_0^\varepsilon) \\ &\le \pi d b^2 |\ln\varepsilon| + \pi d(1-b^2)|\ln\delta| + C = \pi d b^2 |\ln\xi| + \pi d|\ln\delta| + C. \end{aligned}$$

Now we prove the second assertion: when $M < d$ we have

$$\inf_{v \in H^1_g(\Omega)} F_\varepsilon(v, \Omega) \leq \pi d b^2 |\ln \xi| + \pi \sum_i d_i^2 |\ln \delta| + C.$$

Let $d_1, ..., d_M \in \mathbb{N}$ be s.t. $\sum d_i = d$. Set $c_0 = 10^{-2d} \cdot \mathrm{dist}(0, \partial\omega)$. For $i \in \{1, ..., M\}$ s.t. $d_i > 0$, fix $\alpha_{1,i}, ..., \alpha_{d_i,i} \in B(0, 10^d c_0) \subset \omega$ s.t.

$$\min\left(\min_{j \neq k} |\alpha_{j,i} - \alpha_{k,i}|, \mathrm{dist}(\alpha_{j,i}, \partial\omega)\right) > 4c_0.$$

Consider an ε-dependent map $\tilde{v}_0^\varepsilon \in H^1(\Omega \setminus \cup_{d_i > 0} \overline{B(a_i, 10^d c_0 \delta)}, \mathbb{S}^1)$ s.t.

$$\begin{cases} \tilde{v}_0^\varepsilon = g & \text{on } \partial\Omega \\ \tilde{v}_0^\varepsilon(x) = \dfrac{(x - a_i)^{d_i}}{|x - a_i|^{d_i}} & \text{on } \partial B(a_i, 10^d c_0 \delta) \end{cases}$$

and satisfying

$$\int_{\Omega \setminus \cup_{d_i > 0} \overline{B(a_i, 10^d c_0 \delta)}} |\nabla \tilde{v}_0^\varepsilon|^2 \leq \pi \sum d_i^2 |\ln \delta| + C$$

with C depending only on Ω, ω and g.

(Such maps do exist, *e.g.*, consider the map introduced in [11], Remark I.5.)

For $i \in \{1, ..., M\}$ s.t. $d_i > 0$, we consider a map $v_i^\varepsilon \in H^1(B(0, 10^d c_0) \setminus \cup_{j=1}^{d_i} \overline{B(\alpha_{j,i}, \xi)}, \mathbb{S}^1)$ s.t.

- $v_i^\varepsilon(x) = x^{d_i}/|x|^{d_i}$ on $\partial B(0, 10^d c_0)$,
- $v_i^\varepsilon(x) = (x - \alpha_{j,i})/|x - \alpha_{j,i}|$ on $\partial B(\alpha_{j,i}, \xi)$,
- $\displaystyle\int_{B(0,10^d c_0) \setminus \cup_{j=1}^{d_i} \overline{B(\alpha_{j,i}, \xi)}} |\nabla v_i^\varepsilon|^2 \leq \pi d_i |\ln \xi| + C$ with C depending only on ω.

(For example, the map considered in Remark I.5 in [11] has these properties).

The necessary test function that satisfies the bound (12) is obtained by rescaling the v_i^ε's (in order to have maps defined in balls of size δ) and gluing the rescaled maps with $\tilde{v}_0^\varepsilon$.

C Proof of the η-ellipticity Lemma

The main argument in the proof of the η-ellipticity result is the following convexity lemma which is a generalization of Lemma 8 in [9]. The proof of Lemma 11 is given in [15].

Lemma 11. *[Convexity Lemma]*

Let C be a chord in the closed unit disc, C different from a diameter. Let S be the smallest of two regions enclosed by the chord and the boundary of the disc.

Let O be a Lipschitz, bounded, connected domain and let $g \in C(\partial O, S)$.

Assume that v minimizes Ginzburg-Landau type energy

$$\tilde{F}(v) = \int_O \left\{\tilde{\alpha}(x)|\nabla v|^2 + \tilde{\beta}(x)(1 - |v|^2)^2\right\} \mathrm{d}x$$

in $H^1_g(O)$, with $\tilde{\alpha}, \tilde{\beta} \in L^\infty(O, \mathbb{R})$ satisfying $\mathrm{essinf}\tilde{\alpha} > 0, \mathrm{essinf}\tilde{\beta} > 0$. *Then $v(O) \subset S$.*

We prove the first part of the lemma 1. Let $x \in \Omega$ be s.t. $\mathrm{dist}(x, \partial\Omega) \geq \varepsilon^{1/4}$. We have

$$\begin{aligned} F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4})) &\geq \frac{b}{2} \int_{B(x,\varepsilon^{1/4})\setminus\overline{B(x,\varepsilon^{1/2})}} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \\ &= \frac{b}{2} \int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} \frac{1}{r} \cdot r \int_{\partial B(x,r)} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\}. \end{aligned}$$

By Mean Value theorem, exists $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$ s.t

$$r \int_{\partial B(x,r)} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \leq \frac{\frac{2}{b} F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4}))}{\frac{1}{4}|\ln \varepsilon|}.$$

There is $C_2 = C_2(\chi, b) > 0$ s.t if $F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4})) \leq \chi^2 |\ln \varepsilon|$, we have

$$\mathrm{Var}(v_\varepsilon, \partial B(x, r)) \leq C_2 \chi, \text{ where } \mathrm{Var}\,(v_\varepsilon, \partial B(x, r)) := \int_{\partial B(x,r)} |\partial_\tau v_\varepsilon|. \tag{117}$$

It follows that

$$|v_\varepsilon|^2 \geq 1 - 3C_2\chi \text{ on } \partial B(x, r). \tag{118}$$

Indeed, arguing by contradiction, assume that there is $\varepsilon_n \downarrow 0$ and $y_n \in \partial B(x, r)$ s.t. $|v_{\varepsilon_n}(y_n)|^2 < 1 - 3C_2\chi$. Using (117) we obtain that

$$|v_{\varepsilon_n}|^2 \leq 1 - C_2\chi \text{ on } \partial B(x, r)$$

which implies that

$$\begin{aligned} 2\pi C_2^2 \chi^2 \varepsilon_n^{2(\frac{1}{2}-1)} &\leq \frac{2\pi C_2^2 r^2 \chi^2}{\varepsilon_n^2} \\ &\leq \frac{r}{\varepsilon_n^2} \int_{\partial B(x,r)} (1 - |v_{\varepsilon_n}|^2)^2 \\ &\leq \frac{\frac{2}{b} F_{\varepsilon_n}(v_{\varepsilon_n}, B(x, {\varepsilon_n}^{1/4}))}{\frac{1}{4}|\ln \varepsilon_n|} \leq \frac{8\chi^2}{b}. \end{aligned}$$

Clearly, the previous assertion gives contradiction.

From (117) and (118), there is $C = C(\chi, b) > 0$ and $\varepsilon_0 = \varepsilon_0(\chi) > 0$ s.t. for $\varepsilon < \varepsilon_0$,

$$v_\varepsilon : \partial B_r \to \{z \in B_1 \,|\, \Re z > 1 - C\chi\}.$$

Using Convexity Lemma (Lemma 11), we find that $|v_\varepsilon| \geq 1 - C\chi$ in $B(x, r) \supset B(x, \varepsilon^{1/2})$.

If $\mathrm{dist}(x, \partial\Omega) < \varepsilon^{1/4}$, we denote $S_r = \Omega \cap \partial B(x, r)$, $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$. Clearly, we have

$$\frac{2}{b} F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4}) \setminus \overline{B(x, \varepsilon^{1/2})}) \geq \int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} \frac{1}{r} \cdot r \,\mathrm{d}r \int_{S_r} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\}.$$

Using mean value argument and the facts that $g_\varepsilon \to g_0$ in $C^1(\partial\Omega, \mathbb{S}^1)$ and that $0 \leq 1 - |g_\varepsilon| \leq \varepsilon$, there are $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$ and $C_1 = C_1(\|g_0\|_{C^1}, \Omega)$ s.t

$$r \int_{\partial(B(x,r)\cap\Omega)} \left\{ |\partial_\tau v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \leq \frac{\frac{2}{b} F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4})) + C_1}{\frac{1}{4}|\ln \varepsilon|}.$$

Using the same argument as before (taking $O = \Omega \cap B(x, r)$) we obtain the desired result.

We prove the second part of the lemma. Let $\mu \in (0,1)$ and $x \in \{\mathrm{dist}(x,\partial\Omega) \geq \varepsilon^{1/4}\}$. Using mean value argument, there is $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$ s.t

$$r\int_{\partial B(x,r)} \left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \leq \frac{\frac{2}{b}F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4}))}{\frac{1}{4}|\ln\varepsilon|}.$$

There exists $C_1 = C_1(\mu, b) > 0$ s.t if $F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) \leq C_1|\ln\varepsilon|$, we have

$$\mathrm{Var}(v_\varepsilon, \partial B(x,r)) \leq \frac{1-\mu}{10} \text{ and } 1-|v_\varepsilon| \leq \frac{1-\mu}{10} \text{ on } \partial B(x,r).$$

By Convexity Lemma $|v_\varepsilon| \geq \mu$ in $B(x,r) \supset B(x,\varepsilon^{1/2})$.

If $\mathrm{dist}(x,\partial\Omega) < \varepsilon^{1/4}$, denote $S_r = \Omega \cap \partial B(x,r)$, $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$. Since

$$\frac{2}{b}F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4}) \setminus \overline{B(x,\varepsilon^{1/2})}) \geq \int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} \frac{1}{r}\cdot r\,\mathrm{d}r \int_{S_r}\left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\}$$

and using the conditions on g_ε, by mean value argument there is $r \in (\varepsilon^{1/2}, \varepsilon^{1/4})$ s.t

$$r\int_{\partial(B(x,r)\cap\Omega)} \left\{|\partial_\tau v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \leq \frac{\frac{2}{b}F_\varepsilon(v_\varepsilon, B(x,\varepsilon^{1/4})) + C(\|g_0\|_{C^1}, \Omega)}{\frac{1}{4}|\ln\varepsilon|}.$$

Using the same argument as before, the statement of the lemma follows.

Acknowledgement. The authors thank Prof. Leonid Berlyand and Prof. Petru Mironescu for useful and stimulating discussions. The work of M. Dos Santos and O. Misiats was partially supported by NSF grant DMS-0708324. Part of this work was done when M. Dos Santos was visiting Penn State University. He is grateful to the Math Department and to Prof. Leonid Berlyand for the hospitality and support of his visit.

References

[1] A. Aftalion, E. Sandier, and S. Serfaty. Pinning Phenomena in the Ginzburg-Landau model of Superconductivity. *J. Math. Pures Appl.*, 80(3):339–372, 2001.

[2] S. Alama and L. Bronsard. Pinning effects and their breakdown for a Ginzburg-Landau model with normal inclusions. *J. Math. Phys.*, 46(9):095102, 39, 2005.

[3] N. André, P. Bauman, and D. Phillips. Vortex pinning with bounded fields for the Ginzburg-Landau equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(4):705–729, 2003.

[4] N. André and I. Shafrir. Asymptotic behavior of minimizers for the Ginzburg-Landau functional with weight. I, II. *Arch. Rational Mech. Anal.*, 142(1):45–73, 75–98, 1998.

[5] H. Aydi and A. Kachmar. Magnetic vortices for a Ginzburg-Landau type energy with discontinuous constraint. II. *Commun. Pure Appl. Anal.*, 8(3):977–998, 2009.

[6] J. Bardeen and M. Stephen. Theory of the Motion of Vortices in Superconductors. *Phys. Rev*, 140(4A):1197–1207, 1965.

[7] P. Bauman, N. Carlson, and D. Phillips. On the zeros of solutions to Ginzburg-Landau type systems. *SIAM J. Math. Anal.*, 24(5):1283–1293, 1993.

[8] L. Berlyand and P. Mironescu. Ginzburg-Landau minimizers in perforated domains with prescribed degrees. preprint available at http://math.univ-lyon1.fr/ mironescu/prepublications.htm, 2006.

[9] L. Berlyand and P. Mironescu. Two-parameter homogenization for a Ginzburg-Landau problem in a perforated domain. *Netw. Heterog. Media*, 3(3):461–487, 2008.

[10] F. Bethuel, H. Brezis, and F. Hélein. Asymptotics for the minimization of a Ginzburg-Landau functional. *Calc. Var. Partial Differential Equations*, 1(2):123–148, 1993.

[11] F. Bethuel, H. Brezis, and F. Hélein. *Ginzburg-Landau Vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston Inc., Boston, MA, 1994.

[12] H. Brezis. Équations de Ginzburg-Landau et singularités. Notes de cours rédigées par Vicenţiu Rădulescu. *http://inf.ucv.ro/ radulescu/articles/coursHB.pdf*, 2001.

[13] H. Brezis. New questions related to the topological degree. In *The Unity of Mathematics*, volume 244 of *Progr. Math.*, pages 137–154, Boston, MA, 2006. Birkhäuser Boston.

[14] M. del Pino and P. Felmer. On the basic concentration estimate for the Ginzburg-Landau equation. *Differ Integr Equat*, 11(5):771–779, 1998.

[15] M. Dos Santos, P. Mironescu, and O. Misiats. The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part I: the zero degree case. *Commun. Contemp. Math.*, to appear.

[16] B. A. Glowacki and M. Majoros. Superconducting-magnetic heterostructures: a method of decreasing AC losses and improving critical current density in multifilamentary conductors. *J. Phys.: Condens. Matter*, 21(25):254206 (10pp), 2009.

[17] D. Larbalestier, A. Gurevich, M. Feldmann, and A. Polyanskii. High- T_c superconducting material for electric power applications. *Nature*, 414:368–377, 2001.

[18] L. Lassoued and P. Mironescu. Ginzburg-landau type energy with discontinuous constraint. *J. Anal. Math.*, 77:1–26, 1999.

[19] C. Lefter and V. Rădulescu. Minimization problems and corresponding renormalized energies. *Differential Integral Equations*, 9(5):903–917, 1996.

[20] C. Lefter and V. Rădulescu. On the Ginzburg-Landau energy with weight. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(2):171–184, 1996.

[21] F. Lin and Q. Du. Ginzburg-Landau vortices, dynamics, pinning and hysteresis. *SIAM J. Math. Anal.*, 28(6):1265–1293, 1997.

[22] N-G. Meyers. An L^p-estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:189–206, 1963.

[23] P. Mironescu. Explicit bounds for solutions to a Ginzburg-Landau type equation. *Rev. Roumaine Math. Pures Appl.*, 41(3-4):263–271, 1996.

[24] C. Morrey. *Multiple integrals in the calculus of variations*, volume 130. Springer, 1966.

[25] P. Newton and G.Chamoun. Vortex lattice theory: A particle interaction perspective. *SIAM Rev.*, 51(3):501–542, 2009.

[26] J. Rubinstein. On the equilibrium position of Ginzburg Landau vortices. *Z. Angew. Math. Phys.*, 46(5):739–751, 1995.

[27] E. Sandier and S. Serfaty. *Vortices in the Magnetic Ginzburg-Landau Model.* Birkhäuser Boston Inc., Boston, MA, 2007.

[28] I. Sigal and F. Ting. Pinning of magnetic vortices by an external potential. *St. Petersburg Math. J.*, 16(1):211–236, 2005.